



胡克不等式及其应用

田景峰 著



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内容提要

胡克不等式是Hölder不等式的精美改进，由于它克服了Hölder不等式在使用时的缺陷，被美国《数学评论》称之为一个“杰出的、非凡的、新的不等式”。正如Hölder不等式是数学各个领域的重要基石一样，胡克不等式也扮演着同样的角色。近年来关于胡克不等式的研究又有了新的进展，本书的目的就是介绍胡克不等式的近期发展概况，这其中包括反向胡克不等式及其性质、胡克不等式及反向胡克不等式的推广及应用等一系列最新的研究成果，并对已有的成果进行系统的总结，从而使该理论进一步系统化，为进一步深化胡克不等式的研究奠定基础。

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前言

不等式在数学的各个领域都起着非常重要的作用,而且在工程技术中也是一个必不可少的基本工具.事实上,自从 20 世纪以来,不等式就一直是一个非常活跃而又有吸引力的研究领域,特别是现在不等式的研究空前活跃,研究的深度和广度都在迅速扩大^[15].

经典的 Hölder 不等式是数学家 Hölder^[20] 于 1889 年给出的如下形式的

$$\sum_{k=1}^n a_k b_k \leq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q}},$$

其中 $a_k \geq 0, b_k \geq 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$ (当 $0 < p < 1$ 时,上式中不等号反向,且 $a_k > 0, b_k > 0$).事实上, Roger^[21] 比 Hölder 早一年得到上式,但习惯上人们一直将上式称为 Hölder 不等式.

众所周知, Hölder 不等式是数学很多领域的重要基石,是深入解决问题的桥梁.自从 Hölder 给出这个不等式以来,对它的研究就没有中断过.著名数学家 Hardy 等在其名著《不等式》中再三强调 Hölder 不等式“极为重要”和“到处都要用到”,这个不等式和 Minkowski 不等式、算术平均与几何平均不等式构成了该文献中前面 6 章的主题,占了全书一半以上的篇幅^[15].一百多年来,出现了大量的关于这个不等式的改进、推广以及应用的文献.

尽管 Hölder 不等式在数学的很多领域有着重要的应用,但是有些问题用 Hölder 不等式估计时往往得不到较为精确的刻画.例如,设

$$a_{2k-1} = b_{2k} = 1, a_{2k} = b_{2k-1} = 0, \quad k = 1, 2, \dots, N, n = 2N,$$

显然 $\sum_{k=1}^n a_k b_k = 0$, 而此时 Hölder 不等式的右端却是 N , 与 0 相差甚远!

基于此, 我国数学家胡克^[24] 于 1981 年在《中国科学》上给出了一个新的不等式, 这个不等式的出现克服了 Hölder 不等式在使用时的缺陷, 美国《数学评论》称之为“一个杰出的、非凡的、新的不等式”^[15]. 国际数学界将这个不等式命名为胡克不等式. 经典的 Hölder 不等式在数学中起着基础性作用并且有着广泛的应用领域, 作为 Hölder 不等式精美改进的胡克不等式在其中的某些领域也扮演着同样的角色. 事实上, 自从胡克给出该不等式以来, 就出现了大量的关于该不等式的研究文献.

近几年来, 对于胡克不等式的研究又有了新的进展. 本书出版的目的除了系统地介绍国内外学者对胡克不等式的研究成果外, 着重叙述作者本人的一系列研究工作.

本书的内容安排如下: 第 1 章预备知识, 主要介绍一些常用的基础不等式以及 Hölder 不等式、Minkowski 不等式的推广; 第 2 章胡克不等式, 主要介绍胡克不等式及其若干推广; 第 3 章反向胡克不等式, 主要介绍反向胡克不等式及其若干推广; 第 4 章几个重要的不等式构成函数的单调性性质, 主要介绍胡克不等式、反向胡克不等式、Hölder 不等式以及 Minkowski 不等式构成函数的单调性性质; 第 5 章应用, 主要介绍胡克不等式及反向胡克不等式的一系列应用.

作者深切怀念已故的著名数学家胡克教授. 为了本书的系统性和完整性, 作者引用了胡克教授所著的《解析不等式的若干问题》的部分内容, 在此, 向他表示衷心的感谢.

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由于作者才疏学浅, 不妥与疏漏之处在所难免, 恳请同仁及读者不吝赐教.

田景峰

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第1章

预备知识

为了方便读者,在这一章中,我们主要给出本书定理的证明中经常要用到的基本不等式以及 Hölder 不等式的重要推广.

1.1 几个常用的基础不等式

定理 1.1 (Cauchy-Schwarz 不等式) 设 a_r, b_r ($r = 1, 2, \dots, n$) 为实数列, 则

$$\left(\sum_{r=1}^n a_r b_r\right)^2 \leq \left(\sum_{r=1}^n a_r^2\right) \left(\sum_{r=1}^n b_r^2\right). \quad (1.1)$$

定理 1.2 (Hölder 不等式) 设 $a_r, b_r \geq 0$ ($r = 1, 2, \dots, n$). 如果 $p \geq q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, 则

$$\sum_{r=1}^n a_r b_r \leq \left(\sum_{r=1}^n a_r^p\right)^{\frac{1}{p}} \left(\sum_{r=1}^n b_r^q\right)^{\frac{1}{q}}; \quad (1.2)$$

如果 $p > 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则有反向不等式

$$\sum_{r=1}^n a_r b_r \geq \left(\sum_{r=1}^n a_r^p\right)^{\frac{1}{p}} \left(\sum_{r=1}^n b_r^q\right)^{\frac{1}{q}}, \quad (1.3)$$

此时要求 $a_r, b_r > 0$.

相应的积分型 Hölder 不等式如下:

定理 1.3 设 $f(x), g(x) \geq 0$. 如果 $p \geq q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, 则

$$\int f(x)g(x)dx \leq \left(\int f^p(x)dx \right)^{\frac{1}{p}} \left(\int g^q(x)dx \right)^{\frac{1}{q}}; \quad (1.4)$$

如果 $p > 0$, $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则有反向不等式

$$\int f(x)g(x)dx \geq \left(\int f^p(x)dx \right)^{\frac{1}{p}} \left(\int g^q(x)dx \right)^{\frac{1}{q}}, \quad (1.5)$$

此时要求 $f(x), g(x) > 0$.

定理 1.4 (Minkowski 不等式) 设 $a_r, b_r \geq 0$ ($r = 1, 2, \dots, n$). 若 $p > 1$, 则

$$\left[\sum_{r=1}^n (a_r + b_r)^p \right]^{\frac{1}{p}} \leq \left(\sum_{r=1}^n a_r^p \right)^{\frac{1}{p}} + \left(\sum_{r=1}^n b_r^p \right)^{\frac{1}{p}}; \quad (1.6)$$

若 $0 < p < 1$, 则有反向不等式

$$\left[\sum_{r=1}^n (a_r + b_r)^p \right]^{\frac{1}{p}} \geq \left(\sum_{r=1}^n a_r^p \right)^{\frac{1}{p}} + \left(\sum_{r=1}^n b_r^p \right)^{\frac{1}{p}}. \quad (1.7)$$

相应的积分型 Minkowski 不等式如下:

定理 1.5 设 $f(x), g(x) \geq 0$. 若 $p > 1$, 则

$$\left[\int (f(x) + g(x))^p dx \right]^{\frac{1}{p}} \leq \left(\int f^p(x) dx \right)^{\frac{1}{p}} + \left(\int g^p(x) dx \right)^{\frac{1}{p}}; \quad (1.8)$$

若 $0 < p < 1$, 则有反向不等式

$$\left[\int (f(x) + g(x))^p dx \right]^{\frac{1}{p}} \geq \left(\int f^p(x) dx \right)^{\frac{1}{p}} + \left(\int g^p(x) dx \right)^{\frac{1}{p}}. \quad (1.9)$$

定理 1.6 (Dresher 不等式) 设 $a_i, b_i \geq 0$ ($i = 1, 2, \dots, n$). 若 $p > 1 > r > 0$, 则

$$\left[\frac{\sum_{i=1}^n (a_i + b_i)^p}{\sum_{i=1}^n (a_i + b_i)^r} \right]^{\frac{1}{p-r}} \leq \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^r} \right)^{\frac{1}{p-r}} + \left(\frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^r} \right)^{\frac{1}{p-r}}. \quad (1.10)$$

定理 1.7 (Jensen 不等式) 设 $\mathbf{a} = (a_1, a_2, \dots, a_n)$, $a_i > 0$, 且 $t_r(\mathbf{a}) =$

$$\left(\sum_{i=1}^n a_i^r \right)^{\frac{1}{r}} \quad (r \neq 0), \text{ 则对于 } 0 < r < s, r < s < 0 \text{ 或 } s < 0 < r \text{ 有}$$

$$t_s(\mathbf{a}) < t_r(\mathbf{a}). \quad (1.11)$$

定理 1.8^[19] 如果 $x > -1$, $\alpha > 1$ 或者 $\alpha < 0$, 则有

$$(1+x)^\alpha \geq 1 + \alpha x. \quad (1.12)$$

当 $0 < \alpha < 1$ 时, 上述不等式反向.

定理 1.9^[22] 如果 $x_i \geq 0$, $\lambda_i > 0$, $i = 1, 2, \dots, n$, $0 < p \leq 1$, 则有

$$\sum_{i=1}^n \lambda_i x_i^p \leq \left(\sum_{i=1}^n \lambda_i \right)^{1-p} \left(\sum_{i=1}^n \lambda_i x_i \right)^p. \quad (1.13)$$

当 $p \geq 1$ 或者 $p < 0$ 时, 上述不等式反向.

定理 1.10^[19] 设 $X, Y \geq 0$. 如果 $0 \leq \alpha \leq 1$, 则有

$$X^\alpha Y^{1-\alpha} \leq \alpha X + (1-\alpha)Y. \quad (1.14)$$

当 $\alpha > 1$ 或者 $\alpha < 0$ 时, 上述不等式反向.

1.2 Hölder 不等式及 Minkowski 不等式的推广

这一节我们将给出 Hölder 不等式及 Minkowski 不等式的几个重要的、常用的推广.

首先, 给出 Hölder 不等式的如下推广:

定理 1.11^[38] 设 $A_{ij} \geq 0$ ($i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$).

(1) 如果 β_j 是正数, 并且 $\sum_{j=1}^m \frac{1}{\beta_j} \geq 1$, 则有

$$\sum_{i=1}^n \prod_{j=1}^m A_{ij} \leq \prod_{j=1}^m \left(\sum_{i=1}^n A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}}. \quad (1.15)$$

(2) 如果 $\beta_1 > 0$, $\beta_j < 0$ ($j = 2, 3, \dots, m$), 并且 $\sum_{j=1}^m \frac{1}{\beta_j} \leq 1$, 则有

$$\sum_{i=1}^n \prod_{j=1}^m A_{ij} \geq \prod_{j=1}^m \left(\sum_{i=1}^n A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}}. \quad (1.16)$$

(3) 如果 $\beta_j < 0$ ($j = 1, 2, \dots, m$), 则有

$$\sum_{i=1}^n \prod_{j=1}^m A_{ij} \geq \prod_{j=1}^m \left(\sum_{i=1}^n A_{ij}^{\beta_j} \right)^{\frac{1}{\beta_j}}. \quad (1.17)$$

定理 1.12^[34] 设 $a_{rj} > 0$ ($r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$), $\lambda_1 \neq 0$, $\lambda_j < 0$

($j = 2, 3, \dots, m$), 并且 $\tau = \max \left\{ \sum_{j=1}^m \frac{1}{\lambda_j}, 1 \right\}$, 则有

$$\sum_{r=1}^n \prod_{j=1}^m a_{rj} \geq n^{1-\tau} \prod_{j=1}^m \left(\sum_{r=1}^n a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \quad (1.18)$$

证 (1) 当 $\lambda_1 < 0$ 时, 显然 $\tau = 1$. 此时不等式 (1.18) 就是不等式 (1.17).

(2) 当 $\lambda_1 > 0$ 并且 $\sum_{j=1}^m \frac{1}{\lambda_j} \geq 1$ 时, 记 $\sum_{j=1}^m \frac{1}{\lambda_j} = t$ ($t \geq 1$), 则有

$\sum_{j=1}^m \frac{1}{t\lambda_j} = 1$. 由不等式 (1.16) 可知

$$\begin{aligned} \left(\sum_{r=1}^n \prod_{j=1}^m a_{rj} \right)^2 &= \sum_{s=1}^n \left(\prod_{i=1}^m a_{si} \right) \sum_{r=1}^n \prod_{j=1}^m a_{rj} \\ &\geq \sum_{s=1}^n \left(\prod_{i=1}^m a_{si} \right) \left[\prod_{j=1}^m \left(\sum_{r=1}^n a_{rj}^{t\lambda_j} \right)^{\frac{1}{t\lambda_j}} \right] \end{aligned}$$

$$\begin{aligned}
 &= \sum_{s=1}^n \left\{ \left(a_{s1}^{t\lambda_1} \sum_{r=1}^n a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_1} - \sum_{j=2}^m \frac{1}{t\lambda_j}} \cdot \left[\prod_{j=2}^m \left(a_{s1}^{t\lambda_1} \sum_{r=1}^n a_{rj}^{t\lambda_j} \right)^{\frac{1}{t\lambda_j}} \right] \right. \\
 &\quad \cdot \left. \left[\prod_{j=2}^m \left(a_{sj}^{t\lambda_j} \sum_{r=1}^n a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_j}} \right] \right\}. \tag{1.19}
 \end{aligned}$$

进而根据

$$\left(\frac{1}{t\lambda_1} - \sum_{j=2}^m \frac{1}{t\lambda_j} \right) + \frac{1}{t\lambda_2} + \frac{1}{t\lambda_3} + \cdots + \frac{1}{t\lambda_m} + \frac{1}{t\lambda_2} + \frac{1}{t\lambda_3} + \cdots + \frac{1}{t\lambda_m} = 1.$$

对不等式(1.19)的右端利用(1.16)可得

$$\begin{aligned}
 \left(\sum_{r=1}^n \prod_{j=1}^m a_{rj} \right)^2 &\geq \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{t\lambda_1} a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_1} - \sum_{j=2}^m \frac{1}{t\lambda_j}} \\
 &\quad \cdot \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{t\lambda_1} a_{rj}^{t\lambda_j} \right)^{\frac{1}{t\lambda_j}} \right] \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{sj}^{t\lambda_j} a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_j}} \right]. \tag{1.20}
 \end{aligned}$$

此外, 利用定理 1.9, 我们有

$$\begin{aligned}
 &\left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{t\lambda_1} a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_1} - \sum_{j=2}^m \frac{1}{t\lambda_j}} \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{t\lambda_1} a_{rj}^{t\lambda_j} \right)^{\frac{1}{t\lambda_j}} \right] \\
 &\quad \cdot \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{sj}^{t\lambda_j} a_{r1}^{t\lambda_1} \right)^{\frac{1}{t\lambda_j}} \right] \\
 &\geq (n^2)^{(1-t)} \left(\frac{1}{t\lambda_1} - \sum_{j=2}^m \frac{1}{t\lambda_j} \right) \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{\lambda_1} a_{r1}^{\lambda_1} \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \\
 &\quad \cdot \left[\prod_{j=2}^m (n^2)^{\frac{1-t}{t\lambda_j}} \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{\lambda_1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \\
 &\quad \cdot \left[\prod_{j=2}^m (n^2)^{\frac{1-t}{t\lambda_j}} \left(\sum_{s=1}^n \sum_{r=1}^n a_{sj}^{\lambda_j} a_{r1}^{\lambda_1} \right)^{\frac{1}{\lambda_j}} \right] \\
 &= (n^2)^{1-t} \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{\lambda_1} a_{r1}^{\lambda_1} \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \\
 &\quad \cdot \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{\lambda_1} a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n a_{sj}^{\lambda_j} a_{r1}^{\lambda_1} \right)^{\frac{1}{\lambda_j}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= n^{2-2t} \left(\sum_{r=1}^n a_{r1}^{\lambda_1} \right)^{\frac{2}{\lambda_1} - \sum_{j=2}^m \frac{2}{\lambda_j}} \\
 &\quad \cdot \left\{ \prod_{j=2}^m \left[\left(\sum_{s=1}^n \sum_{r=1}^n a_{s1}^{\lambda_1} a_{rj}^{\lambda_j} \right) \left(\sum_{s=1}^n \sum_{r=1}^n a_{sj}^{\lambda_j} a_{r1}^{\lambda_1} \right) \right]^{\frac{1}{\lambda_j}} \right\} \\
 &= n^{2-2t} \left(\sum_{r=1}^n a_{r1}^{\lambda_1} \right)^{\frac{2}{\lambda_1} - \sum_{j=2}^m \frac{2}{\lambda_j}} \\
 &\quad \cdot \left\{ \prod_{j=2}^m \left[\left(\sum_{s=1}^n a_{s1}^{\lambda_1} \right) \left(\sum_{r=1}^n a_{rj}^{\lambda_j} \right) \left(\sum_{s=1}^n a_{sj}^{\lambda_j} \right) \left(\sum_{r=1}^n a_{r1}^{\lambda_1} \right) \right]^{\frac{1}{\lambda_j}} \right\} \\
 &= n^{2-2t} \prod_{j=1}^m \left(\sum_{r=1}^n a_{rj}^{\lambda_j} \right)^{\frac{2}{\lambda_j}}. \tag{1.21}
 \end{aligned}$$

联合不等式(1.20)和(1.21)立刻可得我们要证的不等式(1.18).

(3) 当 $\lambda_1 > 0$ 并且 $\sum_{j=1}^m \frac{1}{\lambda_j} \leq 1$ 时, 很显然不等式(1.18)就是不等式(1.16). □

定理 1.13^[40] 设 $a_{rj} > 0$ ($r = 1, 2, \dots, n, j = 1, 2, \dots, m$), $\lambda_j > 0$ ($j = 1, 2, \dots, m$), 并且 $\gamma = \min \left\{ \sum_{j=1}^m \frac{1}{\lambda_j}, 1 \right\}$, 则有

$$\sum_{r=1}^n \prod_{j=1}^m a_{rj} \leq n^{1-\gamma} \prod_{j=1}^m \left(\sum_{r=1}^n a_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}}. \tag{1.22}$$

证 定理 1.13 的证明类似于定理 1.12 的证明, 在此省略. □

接下来, 我们给出 Minkowski 不等式的推广:

定理 1.14^[29] 设 $f_i(x) \geq 0$ ($i = 1, 2, \dots, n$). 若 $p > 1$, 则

$$\left[\int \left(\sum_{i=1}^n f_i(x) \right)^p dx \right]^{\frac{1}{p}} \leq \sum_{i=1}^n \left(\int f_i^p(x) dx \right)^{\frac{1}{p}}; \tag{1.23}$$

若 $0 < p < 1$, 则有反向不等式

$$\left[\int \left(\sum_{i=1}^n f_i(x) \right)^p dx \right]^{\frac{1}{p}} \geq \sum_{i=1}^n \left(\int f_i^p(x) dx \right)^{\frac{1}{p}}. \quad (1.24)$$

上述推广的 Minkowski 不等式的伴随形式不等式如下:

定理 1.15^[29] 设 $f_i(x) \geq 0$ ($i = 1, 2, \dots, n$). 若 $p > 1$, 则

$$\int \left(\sum_{i=1}^n f_i(x) \right)^p dx \geq \sum_{i=1}^n \int f_i^p(x) dx; \quad (1.25)$$

若 $0 < p < 1$, 则有反向不等式

$$\int \left(\sum_{i=1}^n f_i(x) \right)^p dx \leq \sum_{i=1}^n \int f_i^p(x) dx. \quad (1.26)$$

第2章

胡克不等式及其推广

在这一章中,我们主要介绍胡克不等式以及它的三种推广,这三种推广包括:条件的弱化、维数的增加以及该不等式的复数形式.此外,由这些推广我们还得到了推广的 Hölder 不等式的一些有意义的改进.

2.1 胡克不等式

众所周知, Hölder 不等式是数学很多领域的重要基石,是深入解决问题的桥梁.自从 Hölder 给出这个不等式以来,对它的研究就没有中断过.著名数学家 Hardy 等在其名著 [22] 中再三强调 Hölder 不等式“极为重要”和“到处都要用到”,这个不等式和 Minkowski 不等式、算术平均与几何平均不等式构成了文献中前面 6 章的主题,占了全书一半以上的篇幅.一百多年来,出现了大量的关于这个不等式的改进、推广以及应用的文献.

尽管 Hölder 不等式在数学的很多领域有着重要的应用,但是有些问题用 Hölder 不等式估计时往往得不到较为精确的刻画.例如,设

$$a_{2k-1} = b_{2k} = 0, a_{2k} = b_{2k-1} = 1, \quad k = 1, 2, \dots, N, n = 2N,$$

显然 $\sum_{i=1}^n a_i b_i = 0$, 而此时 Hölder 不等式(1.2)的右端为 N , 与 0 相差较大.

基于此,我国数学家胡克于 1981 年在《中国科学》上给出了 Hölder 不等式的一个如下的新的改进.在此,我们称之为胡克不等式.

定理 2.1^[1] 设 $A_r, B_r \geq 0$, $1 - e_r + e_s \geq 0$ ($r, s = 1, 2, \dots$). 如果 $q \geq p > 0$,

$\frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\begin{aligned} \sum_r A_r B_r &\leq \left(\sum_r A_r^p \right)^{\frac{1}{p} - \frac{1}{q}} \left\{ \left[\left(\sum_r A_r^p \right) \left(\sum_r B_r^q \right) \right]^2 \right. \\ &\quad \left. - \left[\left(\sum_r A_r^p e_r \right) \left(\sum_r B_r^q \right) - \left(\sum_r A_r^p \right) \left(\sum_r B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (2.1)$$

证 下面分两种情况对这个定理进行证明.

(1) 当 $q > p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$ 时, 经过一些简单的运算, 有

$$\begin{aligned} \sum_r A_r B_r \sum_s A_s B_s (1 - e_r + e_s) &= \sum_s \sum_r A_r B_r A_s B_s - \sum_s \sum_r A_r B_r A_s B_s e_r + \sum_s \sum_r A_r B_r A_s B_s e_s \\ &= \left(\sum_r A_r B_r \right)^2. \end{aligned} \quad (2.2)$$

考虑到推广的 Hölder 不等式(1.15), 有

$$\begin{aligned} \sum_r A_r B_r \sum_s A_s B_s (1 - e_r + e_s) &= \sum_r A_r B_r \sum_s A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\ &\leq \sum_r A_r B_r \left(\sum_s A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p}} \left(\sum_s B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\ &= \sum_r \left[\left(\sum_s A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_s A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \right. \\ &\quad \left. \cdot \left(\sum_s B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \right]. \end{aligned} \quad (2.3)$$

由于 $\left(\frac{1}{p} - \frac{1}{q} \right) + \frac{1}{q} + \frac{1}{q} = 1$, 进而在不等式 (2.3) 的右端利用不等式 (1.15), 可得

$$\begin{aligned}
 & \sum_r A_r B_r \sum_s A_s B_s (1 - e_r + e_s) \\
 & \leq \left(\sum_r \sum_s A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_r \sum_s A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & \quad \cdot \left(\sum_r \sum_s B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = \left(\sum_r A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left[\left(\sum_r A_r^p \sum_{s=1}^n B_s^q - \sum_r A_r^p e_r \sum_{s=1}^n B_s^q + \sum_r A_r^p \sum_s B_s^q e_s \right) \right. \\
 & \quad \cdot \left. \left(\sum_r B_r^q \sum_s A_s^p - \sum_r B_r^q e_r \sum_s A_s^p + \sum_r B_r^q \sum_s A_s^p e_s \right) \right]^{\frac{1}{q}} \\
 & = \left(\sum_r A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_r A_r^p \right) \left(\sum_r B_r^q \right) \right]^2 \right. \\
 & \quad \left. - \left[\left(\sum_r A_r^p e_r \right) \left(\sum_r B_r^q \right) - \left(\sum_r A_r^p \right) \left(\sum_r B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (2.4)
 \end{aligned}$$

联合不等式(2.2)和(2.4), 立刻得到要证的不等式(2.1).

(2) 当 $p = q$ 时定理是显然的. □

评注 2.1 接下来我们看看本章开始时举的特例:

$$a_{2k-1} = b_{2k} = 0, \quad a_{2k} = b_{2k-1} = 1, \quad k = 1, 2, \dots, N, \quad n = 2N.$$

如果在(2.1)中取 $e_{2k-1} = 0, e_{2k} = 1, k = 1, 2, \dots, N$, 则由(2.1)可得 $0 \leq 0$. 可见, 不等式(2.1)较 Hölder 不等式刻画这个问题更为精细.

由定理 2.1 和定理 1.8, 很容易得到 Hölder 不等式的如下改进形式:

推论 2.1 设 $A_r > 0, B_r > 0 (r = 1, 2, \dots), 1 - e_r + e_s \geq 0 (r, s = 1, 2, \dots)$,

并且 $q \geq p > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\sum_r A_r B_r \leq \left(\sum_r A_r^p \right)^{\frac{1}{p}} \left(\sum_r B_r^q \right)^{\frac{1}{q}} \left[1 - \frac{1}{2q} \left(\frac{\sum_r B_r^q e_r}{\sum_r B_r^q} - \frac{\sum_r A_r^p e_r}{\sum_r A_r^p} \right)^2 \right]. \quad (2.5)$$

按照类似的证明思路, 我们可以有如下的积分型胡克不等式:

定理 2.2^[1] 设 $f(x), g(x) \geq 0$, $f^p(x), g^q(x)$ 是可积函数, $1 - e(x) + e(y) \geq 0$.

如果 $q \geq p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则

$$\begin{aligned} \int f(x)g(x)dx &\leq \left(\int f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int f^p(x)dx \int g^q(x)dx \right)^2 \right. \\ &\quad - \left(\int f^p(x)e(x)dx \int g^q(x)dx \right. \\ &\quad \left. \left. - \int f^p(x)dx \int g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \end{aligned} \quad (2.6)$$

由定理 2.2 易得如下形式的 Hölder 不等式的改进:

推论 2.2 设 $f(x), g(x) \geq 0$, $f^p(x), g^q(x)$ 是可积函数, $1 - e(x) + e(y) \geq 0$.

如果 $q \geq p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则

$$\begin{aligned} \int f(x)g(x)dx &\leq \left(\int f^p(x)dx \right)^{\frac{1}{p}} \left(\int g^q(x)dx \right)^{\frac{1}{q}} \\ &\quad \cdot \left[1 - \frac{1}{2q} \left(\frac{\int f^p(x)e(x)dx}{\int f^p(x)dx} - \frac{\int g^q(x)e(x)dx}{\int g^q(x)dx} \right)^2 \right]. \end{aligned} \quad (2.7)$$

2.2 胡克不等式的第一种推广

胡克不等式有很多有意义的推广, 在此, 我们先来介绍它的第一种推广, 也就是限制条件弱化的推广. 此外, 由该推广我们很容易得到经典的 Hölder 不等式的推广和一种有意义的改进.

定理 2.3^[41] 设 $A_r \geq 0$, $B_r > 0$ ($r = 1, 2, \dots, n$), $1 - e_r + e_s \geq 0$ ($r, s = 1$,

$2, \dots, n)$, 并且设 $p \geq q > 0$, $\rho = \min \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$, 则有

$$\begin{aligned} \sum_{r=1}^n A_r B_r &\leq n^{1-\rho} \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\ &\quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (2.8)$$

证 我们分两种情况对这个定理进行证明.

(1) 当 $\frac{1}{p} + \frac{1}{q} \geq 1$ 时, 由不等式(1.13)可得

$$\begin{aligned} \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} &= \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\ &\geq \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1 - \frac{1}{p} - \frac{1}{q}} \\ &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s (1 - e_r + e_s) \right)^{\frac{1}{p} + \frac{1}{q}} \\ &= \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1 - \frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right. \\ &\quad \left. - \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s e_r + \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s e_s \right)^{\frac{1}{p} + \frac{1}{q}} \\ &= \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1 - \frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{\frac{1}{p} + \frac{1}{q}} \\ &= \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \\ &= \left(\sum_{r=1}^n A_r B_r \right)^2. \end{aligned} \quad (2.9)$$

考虑到推广的 Hölder 不等式(1.15), 有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\
 & \leq \sum_{r=1}^n A_r B_r \left(\sum_{s=1}^n A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p}} \left(\sum_{s=1}^n B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = \sum_{r=1}^n \left[\left(\sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \right. \\
 & \quad \cdot \left. \left(\sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \right]. \tag{2.10}
 \end{aligned}$$

由于 $\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q} + \frac{1}{q} \geq 1$, 进而在不等式(2.10)的右端利用不等式(1.15)可得

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\
 & \leq \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left[\left(\sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q - \sum_{r=1}^n A_r^p e_r \sum_{s=1}^n B_s^q + \sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q e_s \right) \right. \\
 & \quad \cdot \left. \left(\sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p - \sum_{r=1}^n B_r^q e_r \sum_{s=1}^n A_s^p + \sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p e_s \right) \right]^{\frac{1}{q}} \\
 & = \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
 & \quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \tag{2.11}
 \end{aligned}$$

联合不等式(2.9)和(2.11)立刻得到要证的不等式(2.8).

(2) 当 $\frac{1}{p} + \frac{1}{q} < 1$ 时, 设 $\frac{1}{p} + \frac{1}{q} = t$ ($0 < t < 1$), 则 $\frac{1}{pt} + \frac{1}{qt} = 1$. 一方面, 经过一些简单的运算, 有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 &= \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s - \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_r \\
 & \quad + \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_s \\
 &= \left(\sum_{k=1}^n A_k B_k \right)^2.
 \end{aligned} \tag{2.12}$$

另一方面, 由 Hölder 不等式和(1.15), 有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 &= \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{pt} + \frac{1}{qt}} \\
 &\leq \sum_{r=1}^n A_r B_r \left[\left(\sum_{s=1}^n A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt}} \right. \\
 & \quad \cdot \left. \left(\sum_{s=1}^n B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \right] \\
 &= \sum_{r=1}^n \left[\left(\sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \right. \\
 & \quad \cdot \left(\sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 & \quad \cdot \left. \left(\sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \right] \\
 &\leq \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}}.
 \end{aligned} \tag{2.13}$$

此外, 考虑到 $0 < t < 1$, 利用定理 1.9, 可得

$$\begin{aligned}
 & \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \left(\sum_{r=1}^n \sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 & \leq \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{(1-t) \left(\frac{1}{pt} - \frac{1}{qt} \right)} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{\frac{1-t}{qt}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{\frac{1-t}{qt}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{1-t} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = n^{2-2t} \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p - \sum_{r=1}^n B_r^q e_r \sum_{s=1}^n A_s^p + \sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p e_s \right)^{\frac{1}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q - \sum_{r=1}^n A_r^p e_r \sum_{s=1}^n B_s^q + \sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q e_s \right)^{\frac{1}{q}} \\
 & = n^{2(1-\frac{1}{p}-\frac{1}{q})} \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
 & \quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (2.14)
 \end{aligned}$$

联合不等式(2.12), (2.13)和(2.14), 立刻得到要证的不等式(2.8). \square

利用定理 2.3 和定理 1.8, 很容易得到 Hölder 不等式的如下改进形式:

推论 2.3 设 $A_r, B_r > 0$ ($r = 1, 2, \dots, n$), $1 - e_r + e_s \geq 0$ ($r, s = 1, 2, \dots, n$),

并且 $p \geq q > 0$, $\rho = \min \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$, 则有

$$\sum_{r=1}^n A_r B_r \leq n^{1-\rho} \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} \cdot \left[1 - \frac{1}{2q} \left(\frac{\sum_{r=1}^n B_r^q e_r}{\sum_{k=1}^n B_r^q} - \frac{\sum_{r=1}^n A_r^p e_r}{\sum_{r=1}^n A_r^p} \right)^2 \right]. \quad (2.15)$$

接下来我们给出积分型胡克不等式的第一种推广.

定理 2.4^[41] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) \geq 0$,

$g(x) > 0$, $1 - e(x) + e(y) \geq 0$. 如果 $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} \leq 1$, 则

$$\begin{aligned} & \int_a^b f(x)g(x)dx \\ & \leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \right)^2 \right. \\ & \quad \left. - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \end{aligned} \quad (2.16)$$

证 对于任意的正整数 n , 我们对区间 $[a, b]$ 进行 n 等分:

$$a < a + \frac{b-a}{n} < \dots < a + \frac{b-a}{n}k < \dots < a + \frac{b-a}{n}(n-1) < b,$$

$$x_k = a + \frac{b-a}{n}k, \quad \Delta x_k = \frac{b-a}{n}, \quad k = 0, 1, 2, \dots, n.$$

由定理 2.3 可得

$$\begin{aligned}
\sum_{k=1}^n f(x_k)g(x_k) &\leq n^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k) \right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n f^p(x_k) \right) \left(\sum_{k=1}^n g^q(x_k) \right) \right]^2 \right. \\
&\quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k) \right) \left(\sum_{k=1}^n g^q(x_k) \right) \right. \\
&\quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k) \right) \left(\sum_{k=1}^n g^q(x_k)e(x_k) \right) \right]^2 \right\}^{\frac{1}{2q}}, \quad (2.17)
\end{aligned}$$

也就是

$$\begin{aligned}
\sum_{k=1}^n f(x_k)g(x_k)\frac{b-a}{n} &\leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k)\frac{b-a}{n} \right)^{\frac{1}{p}-\frac{1}{q}} \\
&\quad \cdot \left\{ \left[\left(\sum_{k=1}^n f^p(x_k)\frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k)\frac{b-a}{n} \right) \right]^2 \right. \\
&\quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k)\frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k)\frac{b-a}{n} \right) \right. \\
&\quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k)\frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k)e(x_k)\frac{b-a}{n} \right) \right]^2 \right\}^{\frac{1}{2q}}. \quad (2.18)
\end{aligned}$$

因为 $f(x), g(x), e(x)$ 在 $[a, b]$ 上是正的黎曼可积函数, 于是 $f^p(x), g^q(x), g^q(x)e(x)$ 在 $[a, b]$ 也是可积的. 对不等式(2.18)两端令 $n \rightarrow \infty$, 则有不等式(2.16)成立. \square

由定理 2.4 易得如下形式的 Hölder 不等式的改进:

推论 2.4 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) \geq 0$,

$g(x) > 0$, $1 - e(x) + e(y) \geq 0$. 如果 $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} \leq 1$, 则

$$\begin{aligned}
\int_a^b f(x)g(x)dx &\leq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} \\
&\quad \cdot \left[1 - \frac{1}{2q} \left(\frac{\int_a^b f^p(x)e(x)dx}{\int_a^b f^p(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right]. \quad (2.19)
\end{aligned}$$

2.3 胡克不等式的第二种推广

这一节我们将给出胡克不等式在维数增加条件下的推广.

定理 2.5^[32] 设 $A_{nj} \geq 0$, $\sum_n A_{nj}^{\lambda_j} < \infty$ ($j = 1, 2, \dots, k$), $\lambda_1 \geq \lambda_2 \geq \dots \geq$

$\geq \lambda_k > 0$, $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且 $1 - e_n + e_m \geq 0$, $\sum_n |e_n| < \infty$. 如果 k 是偶数, 则有

$$\begin{aligned} \sum_n \prod_{j=1}^k A_{nj} &\leq \prod_{j=1}^{\frac{k}{2}} \left\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \\ &\quad \cdot \left[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right)^2 \right. \\ &\quad \left. - \left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right) \right. \\ &\quad \left. \left. - \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} e_n \right) \right)^2 \right]^{\frac{1}{2\lambda_{2j}}} \right\}; \quad (2.20) \end{aligned}$$

如果 k 是奇数, 则有

$$\begin{aligned} \sum_n \prod_{j=1}^k A_{nj} &\leq \left(\sum_n A_{nk}^{\lambda_k} \right)^{\frac{1}{\lambda_k}} \cdot \prod_{j=1}^{\frac{k-1}{2}} \left\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \\ &\quad \cdot \left[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right)^2 \right. \\ &\quad \left. - \left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right) \right. \\ &\quad \left. \left. - \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} e_n \right) \right)^2 \right]^{\frac{1}{2\lambda_{2j}}} \right\}. \quad (2.21) \end{aligned}$$

相应的积分形式如下:

定理 2.6^[32] 设 $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$, $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且设 E 是可测集,

$F_j(x)$ 是非负可测函数, $\int_E F_j^{\lambda_j}(x) dx < \infty$, $e(x)$ 是可测函数,

$1 - e(x) + e(y) \geq 0$. 如果 k 是偶数, 则有

$$\begin{aligned} \int_E \prod_{j=1}^k F_j(x) dx &\leq \prod_{j=1}^{\frac{k}{2}} \left\{ \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \\ &\quad \cdot \left[\left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) dx \right)^2 \right. \\ &\quad \left. - \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) e(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) dx \right. \right. \\ &\quad \left. \left. - \int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) e(x) dx \right)^2 \right]^{\frac{1}{2\lambda_{2j}}} \Big\}; \quad (2.22) \end{aligned}$$

如果 k 是奇数, 则有

$$\begin{aligned} \int_E \prod_{j=1}^k F_j(x) dx &\leq \left(\int_E F_k^{\lambda_k}(x) dx \right)^{\frac{1}{\lambda_k}} \\ &\quad \cdot \prod_{j=1}^{\frac{k-1}{2}} \left\{ \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \\ &\quad \cdot \left[\left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) dx \right)^2 \right. \\ &\quad \left. - \left(\int_E F_{2j-1}^{\lambda_{2j-1}}(x) e(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) dx \right. \right. \\ &\quad \left. \left. - \int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx \int_E F_{2j}^{\lambda_{2j}}(x) e(x) dx \right)^2 \right]^{\frac{1}{2\lambda_{2j}}} \Big\}. \quad (2.23) \end{aligned}$$

证 这里仅仅给出离散形式的证明, 读者可以类似地给出积分形式的证明. 经过一些简单的运算, 我们有

$$\begin{aligned}
 & \sum_n \left(\prod_{j=1}^k A_{nj} \right) \sum_m \left(\prod_{i=1}^k A_{mi} \right) (1 - e_n + e_m) \\
 &= \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) - \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) e_n \\
 & \quad + \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) e_m \\
 &= \left(\sum_n \prod_{j=1}^k A_{nj} \right)^2.
 \end{aligned} \tag{2.24}$$

这里我们分两种情况分别给出这个定理的证明.

(1) 当 k 是偶数时, 由不等式(1.15), 有

$$\begin{aligned}
 & \sum_n \left(\prod_{j=1}^k A_{nj} \right) \sum_m \left(\prod_{i=1}^k A_{mi} \right) (1 - e_n + e_m) \\
 &= \sum_n \left(\prod_{j=1}^k A_{nj} \right) \sum_m \prod_{i=1}^k A_{mi} (1 - e_n + e_m)^{\frac{1}{\lambda_i}} \\
 &\leq \sum_n \left(\prod_{j=1}^k A_{nj} \right) \left[\prod_{i=1}^k \left(\sum_m A_{mi}^{\lambda_i} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_i}} \right] \\
 &= \sum_n \left\{ \prod_{j=1}^{\frac{k}{2}} \left[\left(A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \right. \\
 & \quad \cdot \left. \left(A_{n(2j)}^{\lambda_{2j-1}} \sum_m A_{m(2j)}^{\lambda_{2j}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j}}} \right. \\
 & \quad \cdot \left. \left. \left(A_{n(2j)}^{\lambda_{2j}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j}}} \right] \right\}.
 \end{aligned} \tag{2.25}$$

考虑到

$$\begin{aligned}
 & \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2} \right) + \frac{1}{\lambda_2} + \frac{1}{\lambda_2} + \left(\frac{1}{\lambda_3} - \frac{1}{\lambda_4} \right) + \frac{1}{\lambda_4} + \frac{1}{\lambda_4} + \cdots \\
 & \quad + \left(\frac{1}{\lambda_{k-1}} - \frac{1}{\lambda_k} \right) + \frac{1}{\lambda_k} + \frac{1}{\lambda_k} = 1,
 \end{aligned}$$

从而对不等式(2.25)的右边利用(1.15), 有

$$\begin{aligned}
 & \sum_n \left(\prod_{j=1}^k A_{nj} \right) \sum_m \left(\prod_{i=1}^k A_{mi} \right) (1 - e_n + e_m) \\
 & \leq \prod_{j=1}^{\frac{k}{2}} \left[\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j-1}} - \frac{1}{\lambda_{2j}}} \right. \\
 & \quad \cdot \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j)}^{\lambda_{2j}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j}}} \\
 & \quad \cdot \left. \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} (1 - e_n + e_m) \right)^{\frac{1}{\lambda_{2j}}} \right] \\
 & = \prod_{j=1}^{\frac{k}{2}} \left\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{2}{\lambda_{2j-1}} - \frac{2}{\lambda_{2j}}} \right. \\
 & \quad \cdot \left[\left(\sum_n \sum_m A_{n(2j-1)}^{\lambda_{2j-1}} A_{m(2j)}^{\lambda_{2j}} (1 - e_n + e_m) \right) \right. \\
 & \quad \cdot \left. \left(\sum_n \sum_m A_{n(2j)}^{\lambda_{2j}} A_{m(2j-1)}^{\lambda_{2j-1}} (1 - e_n + e_m) \right) \right]^{\frac{1}{\lambda_{2j}}} \Big\} \\
 & = \prod_{j=1}^{\frac{k}{2}} \left\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{2}{\lambda_{2j-1}} - \frac{2}{\lambda_{2j}}} \left[\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j)}^{\lambda_{2j}} \right. \right. \right. \\
 & \quad \left. \left. - \sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \sum_m A_{m(2j)}^{\lambda_{2j}} + \sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \sum_m A_{m(2j)}^{\lambda_{2j}} e_m \right) \right. \\
 & \quad \cdot \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} - \sum_n A_{n(2j)}^{\lambda_{2j}} e_n \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} \right. \\
 & \quad \left. \left. + \sum_n A_{n(2j)}^{\lambda_{2j}} \sum_m A_{m(2j-1)}^{\lambda_{2j-1}} e_m \right) \right]^{\frac{1}{\lambda_{2j}}} \Big\} \\
 & = \prod_{j=1}^{\frac{k}{2}} \left\{ \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right)^{\frac{2}{\lambda_{2j-1}} - \frac{2}{\lambda_{2j}}} \left[\left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right)^2 \right. \right. \\
 & \quad \left. \left. - \left(\left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} \right) \right. \right. \right. \\
 & \quad \left. \left. \left. - \left(\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} \right) \left(\sum_n A_{n(2j)}^{\lambda_{2j}} e_n \right) \right) \right]^2 \right]^{\frac{1}{\lambda_{2j}}} \Big\}. \tag{2.26}
 \end{aligned}$$

由不等式(2.24)和(2.26)立刻得到我们要证的不等式(2.20).

(2) 当 k 是奇数时, 采用与(1)类似的方法易得不等式(2.21). \square

利用上述两个定理, 我们得到一般化的 Hölder 不等式的如下两种改进形式:

推论 2.5 设 $A_{nj} \geq 0$, $0 \neq \sum_n A_{nj}^{\lambda_j} < \infty$ ($j = 1, 2, \dots, k$), $\lambda_1 \geq \lambda_2 \geq \dots \geq$

$\lambda_k > 0$, $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且 $1 - e_n + e_m \geq 0$, $\sum_n |e_n| < \infty$, 则有

$$\sum_n \prod_{j=1}^k A_{nj} \leq \left[\prod_{j=1}^k \left(\sum_n A_{nj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \cdot \left\{ \prod_{j=1}^{\rho(k)} \left[1 - \frac{1}{2\lambda_{2j}} \left(\frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}} \right)^2 \right] \right\}. \quad (2.27)$$

$$\text{其中 } \rho(k) = \begin{cases} \frac{k}{2}, & \text{若 } k \text{ 是偶数,} \\ \frac{k-1}{2}, & \text{若 } k \text{ 是奇数.} \end{cases}$$

推论 2.6 设 $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$, $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且设 E 是可测集,

$F_j(x)$ 是非负可测函数, $0 \neq \int_E F_j^{\lambda_j}(x) dx < \infty$, $e(x)$ 是可测函数,

$1 - e(x) + e(y) \geq 0$, 则有

$$\int_E \prod_{j=1}^k F_j(x) dx \leq \left[\prod_{j=1}^k \left(\int_E F_j^{\lambda_j}(x) dx \right)^{\frac{1}{\lambda_j}} \right] \cdot \left\{ \prod_{j=1}^{\rho(k)} \left[1 - \frac{1}{2\lambda_{2j}} \left(\frac{\int_E F_{2j-1}^{\lambda_{2j-1}}(x) e(x) dx}{\int_E F_{2j-1}^{\lambda_{2j-1}}(x) dx} - \frac{\int_E F_{2j}^{\lambda_{2j}}(x) e(x) dx}{\int_E F_{2j}^{\lambda_{2j}}(x) dx} \right)^2 \right] \right\}. \quad (2.28)$$

$$\text{其中 } \rho(k) = \begin{cases} \frac{k}{2}, & \text{若 } k \text{ 是偶数,} \\ \frac{k-1}{2}, & \text{若 } k \text{ 是奇数.} \end{cases}$$

证 这里只给出推论 2.5 的证明. 推论 2.6 的证明类似. 由不等式 (2.20) 和 (2.21), 有

$$\begin{aligned} \sum_n \prod_{j=1}^k A_{nj} &\leq \left[\prod_{j=1}^k \left(\sum_n A_{nj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \\ &\cdot \left\{ \prod_{j=1}^{\rho(k)} \left[1 - \left(\frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}} \right)^2 \right]^{\frac{1}{2\lambda_{2j}}} \right\}. \end{aligned} \quad (2.29)$$

此外, 经过一些简单的运算, 可以得到

$$\left| \frac{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}} e_n}{\sum_n A_{n(2j-1)}^{\lambda_{2j-1}}} - \frac{\sum_n A_{n(2j)}^{\lambda_{2j}} e_n}{\sum_n A_{n(2j)}^{\lambda_{2j}}} \right| < 1. \quad (2.30)$$

进而, 由定理 1.8、不等式 (2.29) 和 (2.30), 我们得到要证的不等式 (2.27). \square

2.4 胡克不等式的第三种推广

在这一节中, 我们主要介绍胡克不等式的复数形式.

定理 2.7^[13] 设 $a_k, b_k, c_k^{(i)} (k, i = 1, 2, \dots)$ 为复数, e_k 为实数. 设 $r, s > 0$,

$$p, q > 1, \frac{1}{p} + \frac{1}{q} = 1. \text{ 记}$$

$$(\mathbf{a}^r, \mathbf{b}^s) = \sum_{k=1}^n a_k^r \overline{b_k^s}, \|\mathbf{a}\|_p = \sum_{k=1}^n |a_k|^p, \|\mathbf{a}\|_2 = \|\mathbf{a}\|, (|\mathbf{x}|^p, \mathbf{e}) = \sum_{k=1}^n |x_k|^p e_k,$$

如果 $1 - e_k + e_m \geq 0$, 并且 $\|c^{(i)}\| = 1$, 则有

$$|(a, b)| \leq \|a\|_p^{\frac{1}{p}} \|b\|_q^{\frac{1}{q}} (1 - \theta_{m,p}^{(2)}(a, b, c, e))^{\alpha(p)}, \quad (2.31)$$

$$\text{其中, } \beta_p(a, x) = \frac{|(a^{\frac{p}{2}}, x)|}{\|a\|_p^{\frac{1}{2}}}, \quad \lambda_p(a, b, e) = \frac{(|a|^p, e)}{\|a\|_p} - \frac{(|b|^q, e)}{\|b\|_q}, \quad \gamma_p(a, b, c) =$$

$$s_p(a, c) - s_q(b, c), \quad \alpha(p) = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\},$$

$$\begin{aligned} \theta_{m,p}^{(2)}(a, b, c, e) &= \gamma_p^2(a, b, c^{(m)}) + \sum_{i=1}^{m-1} \gamma_p^2(a, b, c^{(i)}) \prod_{k=i+1}^m \beta_p(a, c^{(k)}) \beta_q(b, c^{(k)}) \\ &\quad + \frac{1}{2} \lambda_p^2(a, b, e) \prod_{k=1}^m \beta_p(a, c^{(k)}) \beta_q(b, c^{(k)}). \end{aligned}$$

当 $m=1$ 时, 上式右边 \sum 项为零. 当 $p \neq 2$ 时, $a_k, b_k \geq 0$; 当 $p=2$ 时, a_k, b_k 可为复数.

特别地, 若 $\beta_p(a, c) \beta_q(b, c) < 1$, 则有

$$|(a, b)| \leq \|a\|_p^{\frac{1}{p}} \|b\|_q^{\frac{1}{q}} \left[1 - \frac{1}{1 - \beta_p(a, c) \beta_q(b, c)} (\beta_p(a, c) - \beta_q(b, c))^2 \right]^{\alpha(p)}. \quad (2.32)$$

相应的积分形式如下:

定理 2.8^[13] 设 $f \in L^p(\alpha, \beta)$, $g \in L^q(\alpha, \beta)$, $1 - e(x) + e(y) \geq 0$, $\|c^{(i)}\| = 1$, 则

$$|(f, g)| \leq \|f\|_p^{\frac{1}{p}} \|g\|_q^{\frac{1}{q}} (1 - \theta_{m,p}^{(2)}(f, g, c, e))^{\alpha(p)}, \quad (2.33)$$

$$\text{其中, } \alpha(p) = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad (f^r, g^s) = \int_{\alpha}^{\beta} f^r(x) \overline{g^s(x)} dx, \quad \int_{\alpha}^{\beta} |f|^p dx = \|f\|_p^p,$$

$$\begin{aligned} \theta_{m,p}^{(2)}(f, g, c, e) &= \gamma_p^2(f, g, c^{(m)}) + \sum_{i=1}^{m-1} \gamma_p^2(f, g, c^{(i)}) \prod_{k=i+1}^m \beta_p(f, c^{(k)}) \beta_q(g, c^{(k)}) \\ &\quad + \frac{1}{2} \lambda_p^2(f, g, e) \prod_{k=1}^m \beta_p(f, c^{(k)}) \beta_q(g, c^{(k)}). \end{aligned}$$

当 $m=1$ 时, 此式右边 \sum 项为零. 当 $p \neq 2$ 时, $f, g \geq 0$; 当 $p=2$ 时, f, g 为复数.

特别地, 若 $\beta_p(f, c)\beta_q(g, c) < 1$, 则有

$$|(f, g)| \leq \|f\|_p^{\frac{1}{p}} \|g\|_q^{\frac{1}{q}} \left[1 - \frac{1}{1 - \beta_p(f, c)\beta_q(g, c)} (\beta_p(f, c) - \beta_q(g, c))^2 \right]^{\alpha(p)}. \quad (2.34)$$

在此我们只给出上述离散形式的证明, 相应的积分形式的证明, 读者可以类似地给出. 要证明定理 2.7, 首先证明其 $p = 2$ 时的情形. 下面用数学归纳法来证明.

当 $m = 1$ 时, 由胡克不等式以及定理 1.8 可知

$$\begin{aligned} |(\mathbf{a}, \mathbf{b})| &\leq \|\mathbf{a}\|^{\frac{1}{2}} \|\mathbf{b}\|^{\frac{1}{2}} (1 - \lambda_2^2(\mathbf{a}, \mathbf{b}, \mathbf{e}))^{\frac{1}{4}} \\ &\leq \|\mathbf{a}\|^{\frac{1}{2}} \|\mathbf{b}\|^{\frac{1}{2}} \left(1 - \frac{1}{4} \lambda_2^2(\mathbf{a}, \mathbf{b}, \mathbf{e}) \right). \end{aligned} \quad (2.35)$$

由 Gram 不等式可知

$$\begin{vmatrix} (\mathbf{a}, \mathbf{a}) & (\mathbf{a}, \mathbf{b}) & (\mathbf{a}, \mathbf{c}) \\ (\mathbf{b}, \mathbf{a}) & (\mathbf{b}, \mathbf{b}) & (\mathbf{b}, \mathbf{c}) \\ (\mathbf{c}, \mathbf{a}) & (\mathbf{c}, \mathbf{b}) & (\mathbf{c}, \mathbf{c}) \end{vmatrix} \geq 0. \quad (2.36)$$

如果 $(\mathbf{c}, \mathbf{c}) = 1$, 则(2.36)可变形为

$$\begin{aligned} (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - |(\mathbf{a}, \mathbf{b})|^2 - (\mathbf{a}, \mathbf{a})|(\mathbf{b}, \mathbf{c})|^2 - (\mathbf{b}, \mathbf{b})|(\mathbf{a}, \mathbf{c})|^2 \\ + 2 \operatorname{Re}(\mathbf{a}, \mathbf{b})(\mathbf{c}, \mathbf{a})(\mathbf{b}, \mathbf{c}) \geq 0. \end{aligned} \quad (2.37)$$

结合(2.35)可知

$$|(\mathbf{a}, \mathbf{b})(\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{c})| \leq |(\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{c})| (\|\mathbf{a}\|^{\frac{1}{2}} \|\mathbf{b}\|^{\frac{1}{2}}) \left(1 - \frac{1}{4} \lambda_2^2(\mathbf{a}, \mathbf{b}, \mathbf{e}) \right). \quad (2.38)$$

进而由(2.37)和(2.38)可得

$$\begin{aligned} |(\mathbf{a}, \mathbf{b})|^2 &\leq (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - \left(\|\mathbf{a}\|^{\frac{1}{2}} |(\mathbf{b}, \mathbf{c})| - \|\mathbf{b}\|^{\frac{1}{2}} |(\mathbf{a}, \mathbf{c})| \right)^2 \\ &\quad - \frac{1}{2} |(\mathbf{a}, \mathbf{c})(\mathbf{b}, \mathbf{c})| \|\mathbf{a}\|^{\frac{1}{2}} \|\mathbf{b}\|^{\frac{1}{2}} \lambda_2^2(\mathbf{a}, \mathbf{b}, \mathbf{e}). \end{aligned} \quad (2.39)$$

(2.39)就是定理 2.7 的(2.31)当 $p = 2, m = 1$ 时的情形. 现设 $p = 2, m = k$ 时定理成立, 即有

$$\begin{aligned} |(\mathbf{a}, \mathbf{b})| &\leq (\|\mathbf{a}\| \|\mathbf{b}\|)^{\frac{1}{2}} (1 - \theta_{k,2}^2(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}))^{\frac{1}{2}} \\ &\leq (\|\mathbf{a}\| \|\mathbf{b}\|)^{\frac{1}{2}} \left(1 - \frac{1}{2} \theta_{k,2}^2(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) \right). \end{aligned} \quad (2.40)$$

由(2.36)和(2.40)可知

$$\begin{aligned}
 |(\mathbf{a}, \mathbf{b})|^2 &\leq (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - (\mathbf{a}, \mathbf{a})|(\mathbf{b}, \mathbf{c}^{(k+1)})|^2 - (\mathbf{b}, \mathbf{b})|(\mathbf{a}, \mathbf{c}^{(k+1)})|^2 \\
 &\quad + 2|(\mathbf{a}, \mathbf{b})||(\mathbf{a}, \mathbf{c}^{(k+1)})||(\mathbf{b}, \mathbf{c}^{(k+1)})| \\
 &\leq (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) - \left(\|\mathbf{a}\|^{\frac{1}{2}} |(\mathbf{b}, \mathbf{c}^{(k+1)})| - \|\mathbf{b}\|^{\frac{1}{2}} |(\mathbf{a}, \mathbf{c}^{(k+1)})| \right)^2 \\
 &\quad - \left(\|\mathbf{a}\| \|\mathbf{b}\| \right)^{\frac{1}{2}} |(\mathbf{a}, \mathbf{c}^{(k+1)})(\mathbf{b}, \mathbf{c}^{(k+1)})| \cdot \theta_{k,2}^{(2)}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e}) \\
 &= (\mathbf{a}, \mathbf{a})(\mathbf{b}, \mathbf{b}) (1 - \theta_{k+1,2}^{(2)}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})). \tag{2.41}
 \end{aligned}$$

于是当 $p = 2$ 时定理 2.7 的(2.31)成立.

下面证明 $p \neq 2$ 时的情形. 不妨设 $p > q > 1$. 已知 $a_k, b_k > 0$, 由 Hölder 不等式可知

$$\sum_{k=1}^n a_k b_k = \sum_{k=1}^n a_k b_k^{\frac{q}{p}} b_k^{1-\frac{q}{p}} \leq (\mathbf{a}^{\frac{p}{2}}, \mathbf{b}^{\frac{q}{2}})^{\frac{2}{p}} \|\mathbf{b}\|_q^{1-\frac{2}{p}}. \tag{2.42}$$

进而由定理 2.7 中 $p = 2$ 时的情形可得

$$(\mathbf{a}^{\frac{q}{2}}, \mathbf{b}^{\frac{p}{2}})^2 \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q (1 - \theta_{m,p}^{(2)}(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{e})). \tag{2.43}$$

将(2.43)代入(2.42)即得定理的(2.31)的结论.

由假设 $x = \beta_p(\mathbf{a}, \mathbf{c})\beta_q(\mathbf{b}, \mathbf{c}) < 1$, 知 $x^n \rightarrow 0$ ($n \rightarrow \infty$). 因此, 在(2.31)中取 $c^{(i)} = c$ 即得(2.32)的结论.

第3章

反向胡克不等式及其推广

在这一章中,我们将主要介绍反向胡克不等式及其三种推广形式.此外,由这些推广我们还将得到推广的反向 Hölder 不等式的一些有意义的改进.

3.1 反向胡克不等式^[32]

这一节我们将首先给出离散型反向胡克不等式的理想形式.

定理 3.1 设 $A_r \geq 0, B_r > 0$ ($r = 1, 2, \dots, n$), 并且 $1 - e_r + e_s \geq 0$ ($r, s = 1, 2, \dots, n$). 如果 $q < 0, p > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\begin{aligned} \sum_{r=1}^n A_r B_r \geq & \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p} - \frac{1}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\ & - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) \right. \\ & \left. \left. - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (3.1)$$

证 经过一些简单的运算, 我们有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 &= \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s - \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_r + \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_s \\
 &= \left(\sum_{r=1}^n A_r B_r \right)^2. \tag{3.2}
 \end{aligned}$$

由推广的 Hölder 不等式(1.15), 有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 &= \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\
 &\geq \sum_{r=1}^n A_r B_r \left(\sum_{s=1}^n A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p}} \left(\sum_{s=1}^n B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &= \sum_{r=1}^n \left[\left(\sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \right. \\
 &\quad \cdot \left. \left(\sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \right]. \tag{3.3}
 \end{aligned}$$

由于 $\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{q} + \frac{1}{q} = 1$, 进而在不等式(3.3)的右端利用不等式(1.15)

可得

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 &\geq \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &= \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left[\left(\sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q - \sum_{r=1}^n A_r^p e_r \sum_{s=1}^n B_s^q + \sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q e_s \right) \right. \\
 &\quad \cdot \left. \left(\sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p - \sum_{r=1}^n B_r^q e_r \sum_{s=1}^n A_s^p + \sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p e_s \right) \right]^{\frac{1}{q}}
 \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
&\quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (3.4)
\end{aligned}$$

联合不等式(3.2)和(3.4), 立刻得到要证的不等式(3.1). \square

由定理 3.1 和定理 1.8, 很容易得到 Hölder 不等式的如下改进形式:

推论 3.1 设 $A_r > 0, B_r > 0$ ($r = 1, 2, \dots, n$), 并且 $1 - e_r + e_s \geq 0$ ($r, s = 1, 2, \dots, n$). 如果 $q < 0, p > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\sum_{r=1}^n A_r B_r \geq \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} \left[1 - \frac{1}{2q} \left(\frac{\sum_{r=1}^n B_r^q e_r}{\sum_{k=1}^n B_r^q} - \frac{\sum_{r=1}^n A_r^p e_r}{\sum_{r=1}^n A_r^p} \right)^2 \right]. \quad (3.5)$$

接下来我们给出积分型胡克不等式的反向形式.

定理 3.2 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x), g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则

$$\begin{aligned}
&\int_a^b f(x)g(x)dx \\
&\geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \right)^2 \right. \\
&\quad \left. - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \quad (3.6)
\end{aligned}$$

证 对于任意的正整数 n , 对区间 $[a, b]$ 进行 n 等分:

$$a < a + \frac{b-a}{n} < \dots < a + \frac{b-a}{n}k < \dots < a + \frac{b-a}{n}(n-1) < b,$$

$$x_k = a + \frac{b-a}{n}k, \Delta x_k = \frac{b-a}{n}, \quad k = 0, 1, 2, \dots, n.$$

由定理 3.1 可得

$$\begin{aligned} \sum_{k=1}^n f(x_k)g(x_k) &\geq \left(\sum_{k=1}^n f^p(x_k)\right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n f^p(x_k)\right) \left(\sum_{k=1}^n g^q(x_k)\right) \right]^2 \right. \\ &\quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k)\right) \left(\sum_{k=1}^n g^q(x_k)\right) \right. \\ &\quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k)\right) \left(\sum_{k=1}^n g^q(x_k)e(x_k)\right) \right]^2 \right\}^{\frac{1}{2q}}, \end{aligned} \quad (3.7)$$

也就是

$$\begin{aligned} \sum_{k=1}^n f(x_k)g(x_k) \frac{b-a}{n} &\geq \left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n}\right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n}\right) \left(\sum_{k=1}^n g^q(x_k) \frac{b-a}{n}\right) \right]^2 \right. \\ &\quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k) \frac{b-a}{n}\right) \left(\sum_{k=1}^n g^q(x_k) \frac{b-a}{n}\right) \right. \\ &\quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n}\right) \left(\sum_{k=1}^n g^q(x_k)e(x_k) \frac{b-a}{n}\right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (3.8)$$

因为 $f(x), g(x), e(x)$ 在 $[a, b]$ 上是正的黎曼可积函数, 于是 $f^p(x), g^q(x), g^q(x)e(x)$ 在 $[a, b]$ 上也是可积的. 对不等式(3.8)两端令 $n \rightarrow \infty$, 则有不等式(3.6)成立. \square

由定理 3.2 易得如下形式的 Hölder 不等式的改进:

推论 3.2 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x), g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则

$$\begin{aligned} \int_a^b f(x)g(x)dx &\geq \left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{1}{q}} \\ &\quad \cdot \left[1 - \frac{1}{2q} \left(\frac{\int_a^b f^p(x)e(x)dx}{\int_a^b f^p(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right]. \end{aligned} \quad (3.9)$$

3.2 反向胡克不等式的第一种推广^[36]

这一节我们将给出反向胡克不等式的条件弱化的推广.

定理 3.3 设 $A_r \geq 0, B_r > 0$ ($r = 1, 2, \dots, n$), 并且设 $1 - e_r + e_s \geq 0$

($r, s = 1, 2, \dots, n$). 如果 $q < 0, p > 0, \rho = \max \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$, 则有

$$\begin{aligned} \sum_{r=1}^n A_r B_r \geq n^{1-\rho} \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}-\frac{1}{q}} & \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\ & \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (3.10)$$

证 我们分两种情况对这个定理进行证明.

(1) 当 $\frac{1}{p} + \frac{1}{q} \leq 1$ 时, 经过一些简单的运算, 有

$$\begin{aligned} \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\ = \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s - \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_r + \sum_{s=1}^n \sum_{r=1}^n A_r B_r A_s B_s e_s \\ = \left(\sum_{k=1}^n A_k B_k \right)^2. \end{aligned} \quad (3.11)$$

由不等式(1.13)可得

$$\begin{aligned} \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\ = \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &\leq \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1-\frac{1}{p}-\frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s (1 - e_r + e_s) \right)^{\frac{1}{p}+\frac{1}{q}} \\
 &= \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right. \\
 &\quad \left. - \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s e_r + \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s e_s \right)^{\frac{1}{p}+\frac{1}{q}} \\
 &= \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \right)^{\frac{1}{p}+\frac{1}{q}} \\
 &= \sum_{r=1}^n \sum_{s=1}^n A_r B_r A_s B_s \\
 &= \left(\sum_{r=1}^n A_r B_r \right)^2. \tag{3.12}
 \end{aligned}$$

考虑到推广的 Hölder 不等式(1.15), 有

$$\begin{aligned}
 &\sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p}+\frac{1}{q}} \\
 &\geq \sum_{r=1}^n A_r B_r \left(\sum_{s=1}^n A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p}} \left(\sum_{s=1}^n B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &= \sum_{r=1}^n \left[\left(\sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p}-\frac{1}{q}} \right. \\
 &\quad \cdot \left(\sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &\quad \cdot \left. \left(\sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \right]. \tag{3.13}
 \end{aligned}$$

由于

$$\left(\frac{1}{p} - \frac{1}{q} \right) + \frac{1}{q} + \frac{1}{q} \leq 1,$$

进而在不等式(3.13)的右端利用不等式(1.15) 可得

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{p} + \frac{1}{q}} \\
 & \geq \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & \quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 & = \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left[\left(\sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q - \sum_{r=1}^n A_r^p e_r \sum_{s=1}^n B_s^q + \sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q e_s \right) \right. \\
 & \quad \cdot \left. \left(\sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p - \sum_{r=1}^n B_r^q e_r \sum_{s=1}^n A_s^p + \sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p e_s \right) \right]^{\frac{1}{q}} \\
 & = \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
 & \quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (3.14)
 \end{aligned}$$

联合不等式(3.12)和(3.14), 立刻得到要证的不等式(3.10).

(2) 当 $\frac{1}{p} + \frac{1}{q} \geq 1$ 时, 设 $\frac{1}{p} + \frac{1}{q} = t$ ($t \geq 1$), 则 $\frac{1}{pt} + \frac{1}{qt} = 1$. 由 Hölder

不等式和(1.15), 有

$$\begin{aligned}
 & \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s) \\
 & = \sum_{r=1}^n A_r B_r \sum_{s=1}^n A_s B_s (1 - e_r + e_s)^{\frac{1}{pt} + \frac{1}{qt}} \\
 & \geq \sum_{r=1}^n A_r B_r \left[\left(\sum_{s=1}^n A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt}} \left(\sum_{s=1}^n B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \right] \\
 & = \sum_{r=1}^n \left[\left(\sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \left(\sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \right. \\
 & \quad \cdot \left. \left(\sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &\geq \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \left(\sum_{r=1}^n \sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}}. \tag{3.15}
 \end{aligned}$$

此外, 考虑到 $t \geq 1$, 利用定理 1.9 可得

$$\begin{aligned}
 &\left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{pt} - \frac{1}{qt}} \left(\sum_{r=1}^n \sum_{s=1}^n B_r^{qt} A_s^{pt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^{pt} B_s^{qt} (1 - e_r + e_s) \right)^{\frac{1}{qt}} \\
 &\geq \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{(1-t)(\frac{1}{pt} - \frac{1}{qt})} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{\frac{1-t}{qt}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{\frac{1-t}{qt}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &= \left(\sum_{r=1}^n \sum_{s=1}^n (1 - e_r + e_s) \right)^{1-t} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p A_s^p (1 - e_r + e_s) \right)^{\frac{1}{p} - \frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n \sum_{s=1}^n B_r^q A_s^p (1 - e_r + e_s) \right)^{\frac{1}{q}} \left(\sum_{r=1}^n \sum_{s=1}^n A_r^p B_s^q (1 - e_r + e_s) \right)^{\frac{1}{q}} \\
 &= n^{2-2t} \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p - \sum_{r=1}^n B_r^q e_r \sum_{s=1}^n A_s^p + \sum_{r=1}^n B_r^q \sum_{s=1}^n A_s^p e_s \right)^{\frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q - \sum_{r=1}^n A_r^p e_r \sum_{s=1}^n B_s^q + \sum_{r=1}^n A_r^p \sum_{s=1}^n B_s^q e_s \right)^{\frac{1}{q}}
 \end{aligned}$$

$$\begin{aligned}
 &= n^{2(1-\frac{1}{p}-\frac{1}{q})} \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p}-\frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
 &\quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (3.16)
 \end{aligned}$$

联合不等式(3.11), (3.15)和(3.16), 立刻得到要证的不等式(3.10). \square

由定理 3.3 和定理 1.8, 很容易得到 Hölder 不等式的如下改进形式:

推论 3.3 设 $A_r > 0, B_r > 0$ ($r = 1, 2, \dots, n$), $1 - e_r + e_s \geq 0$ ($r, s = 1, 2, \dots, n$), 并且 $q < 0, p > 0, \rho = \max \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$, 则有

$$\sum_{r=1}^n A_r B_r \geq n^{1-\rho} \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} \left[1 - \frac{1}{2q} \left(\frac{\sum_{r=1}^n B_r^q e_r}{\sum_{k=1}^n B_r^q} - \frac{\sum_{r=1}^n A_r^p e_r}{\sum_{r=1}^n A_r^p} \right)^2 \right]. \quad (3.17)$$

接下来我们给出积分型胡克不等式的第二种推广.

定理 3.4 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x), g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} \geq 1$, 则

$$\begin{aligned}
 &\int_a^b f(x)g(x)dx \\
 &\geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}-\frac{1}{q}} \left[\left(\int_a^b f^p(x)dx \int_a^b g^q(x)dx \right)^2 \right. \\
 &\quad \left. - \left(\int_a^b f^p(x)e(x)dx \int_a^b g^q(x)dx - \int_a^b f^p(x)dx \int_a^b g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{2q}}. \quad (3.18)
 \end{aligned}$$

证 对于任意的正整数 n , 对区间 $[a, b]$ 进行 n 等分:

$$a < a + \frac{b-a}{n} < \dots < a + \frac{b-a}{n}k < \dots < a + \frac{b-a}{n}(n-1) < b,$$

$$x_k = a + \frac{b-a}{n}k, \Delta x_k = \frac{b-a}{n}, \quad k = 0, 1, 2, \dots, n.$$

由定理 3.3 可得

$$\begin{aligned} & \sum_{k=1}^n f(x_k)g(x_k) \\ & \geq n^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k) \right)^{\frac{1}{p}-\frac{1}{q}} \left\{ \left[\left(\sum_{k=1}^n f^p(x_k) \right) \left(\sum_{k=1}^n g^q(x_k) \right) \right]^2 \right. \\ & \quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k) \right) \left(\sum_{k=1}^n g^q(x_k) \right) \right. \\ & \quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k) \right) \left(\sum_{k=1}^n g^q(x_k)e(x_k) \right) \right]^2 \right\}^{\frac{1}{2q}}, \end{aligned} \quad (3.19)$$

也就是

$$\begin{aligned} & \sum_{k=1}^n f(x_k)g(x_k) \frac{b-a}{n} \\ & \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n} \right)^{\frac{1}{p}-\frac{1}{q}} \\ & \quad \cdot \left\{ \left[\left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k) \frac{b-a}{n} \right) \right]^2 \right. \\ & \quad - \left[\left(\sum_{k=1}^n f^p(x_k)e(x_k) \frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k) \frac{b-a}{n} \right) \right. \\ & \quad \left. \left. - \left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n} \right) \left(\sum_{k=1}^n g^q(x_k)e(x_k) \frac{b-a}{n} \right) \right]^2 \right\}^{\frac{1}{2q}}. \end{aligned} \quad (3.20)$$

因为 $f(x), g(x), e(x)$ 在 $[a, b]$ 上是正的黎曼可积函数, 于是 $f^p(x), g^q(x), g^q(x)e(x)$ 在 $[a, b]$ 上也是可积的. 对不等式(3.20)两端令 $n \rightarrow \infty$, 则有不等式(3.18)成立. \square

由定理 3.4 易得如下形式的 Hölder 不等式的改进:

推论 3.4 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$$f(x), g(x) > 0, 1 - e(x) + e(y) \geq 0. \text{ 如果 } q < 0, \frac{1}{p} + \frac{1}{q} \geq 1, \text{ 则}$$

$$\int_a^b f(x)g(x)dx \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} \\ \cdot \left[1 - \frac{1}{2q} \left(\frac{\int_a^b f^p(x)e(x)dx}{\int_a^b f^p(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right]. \quad (3.21)$$

3.3 反向胡克不等式的第二种推广^[32]

这一节我们将给出反向胡克不等式在维数增加情况下的推广.

定理 3.5 设 $A_{rj} > 0$ ($r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$), $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$, 并且 $1 - e_r + e_s \geq 0$ ($s = 1, 2, \dots, n$). 如果 $\lambda_1 > 0$, $\lambda_j < 0$ ($j = 2, 3, \dots, m$), 则

$$\sum_{r=1}^n \prod_{j=1}^m A_{rj} \geq \left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \prod_{j=2}^m \left\{ \left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} \right) \right]^2 \right. \\ \left. - \left[\left(\sum_{r=1}^n A_{r1}^{\lambda_1} e_r \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} \right) \right. \right. \\ \left. \left. - \left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} e_r \right) \right]^2 \right\}^{\frac{1}{2\lambda_j}}. \quad (3.22)$$

相应的积分形式如下:

定理 3.6 设 $F_j(x)$ 是定义在 $[a, b]$ 上的非负可积函数, 并且 $\int_a^b F_j^{\lambda_j}(x)dx$ 存在, 设 $1 - e(x) + e(y) \geq 0$, $\int_a^b e(x)dx < \infty$, 并且 $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$. 如果 $\lambda_1 > 0$, $\lambda_j < 0$ ($j = 2, 3, \dots, m$), 则有

$$\begin{aligned}
 \int_a^b \prod_{j=1}^m F_j(x) dx &\geq \left(\int_a^b F_1^{\lambda_1}(x) dx \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \\
 &\quad \cdot \prod_{j=2}^m \left[\left(\int_a^b F_1^{\lambda_1}(x) dx \int_a^b F_j^{\lambda_j}(x) dx \right)^2 \right. \\
 &\quad \left. - \left(\int_a^b F_1^{\lambda_1}(x) e(x) dx \int_a^b F_j^{\lambda_j}(x) dx \right. \right. \\
 &\quad \left. \left. - \int_a^b F_1^{\lambda_1}(x) dx \int_a^b F_j^{\lambda_j}(x) e(x) dx \right)^2 \right]^{\frac{1}{2\lambda_j}}. \quad (3.23)
 \end{aligned}$$

证 这里只给出定理 3.5 的证明. 定理 3.6 的证明类似. 经过一些简单的运算, 我们有

$$\begin{aligned}
 &\sum_n \left(\prod_{j=1}^k A_{nj} \right) \sum_m \left(\prod_{i=1}^k A_{mi} \right) (1 - e_n + e_m) \\
 &= \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) \\
 &\quad - \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) e_n \\
 &\quad + \sum_n \sum_m \left(\prod_{j=1}^k A_{nj} \right) \left(\prod_{i=1}^k A_{mi} \right) e_m \\
 &= \left(\sum_n \prod_{j=1}^k A_{nj} \right)^2. \quad (3.24)
 \end{aligned}$$

由推广的 Hölder 不等式(1.15)可知

$$\begin{aligned}
 &\sum_{s=1}^n \left(\prod_{i=1}^m A_{si} \right) \sum_{r=1}^n \left(\prod_{j=1}^m A_{rj} \right) (1 - e_r + e_s) \\
 &= \sum_{s=1}^n \left(\prod_{i=1}^m A_{si} \right) \sum_{r=1}^n \prod_{j=1}^m A_{rj} (1 - e_r + e_s)^{\frac{1}{\lambda_j}} \\
 &\geq \sum_{s=1}^n \left(\prod_{i=1}^m A_{si} \right) \left[\prod_{j=1}^m \left(\sum_{r=1}^n A_{rj}^{\lambda_j} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_j}} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{s=1}^n \left\{ \left(A_{s1}^{\lambda_1} \sum_{r=1}^n A_{r1}^{\lambda_1} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \right. \\
 &\quad \cdot \left[\prod_{j=2}^m \left(A_{s1}^{\lambda_1} \sum_{r=1}^n A_{rj}^{\lambda_j} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_j}} \right] \\
 &\quad \cdot \left. \left[\prod_{j=2}^m \left(A_{sj}^{\lambda_j} \sum_{r=1}^n A_{r1}^{\lambda_1} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_j}} \right] \right\}. \quad (3.25)
 \end{aligned}$$

考虑到

$$\left(\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j} \right) + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_m} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} + \cdots + \frac{1}{\lambda_m} = 1,$$

在不等式(3.25)的右边应用不等式(1.15), 可得

$$\begin{aligned}
 &\sum_{s=1}^n \left(\prod_{i=1}^m A_{si} \right) \sum_{r=1}^n \left(\prod_{j=1}^m A_{rj} \right) (1 - e_r + e_s) \\
 &\geq \left(\sum_{s=1}^n \sum_{r=1}^n A_{s1}^{\lambda_1} A_{r1}^{\lambda_1} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_1} - \sum_{j=2}^m \frac{1}{\lambda_j}} \\
 &\quad \cdot \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n A_{s1}^{\lambda_1} A_{rj}^{\lambda_j} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_j}} \right] \\
 &\quad \cdot \left[\prod_{j=2}^m \left(\sum_{s=1}^n \sum_{r=1}^n A_{sj}^{\lambda_j} A_{r1}^{\lambda_1} (1 - e_r + e_s) \right)^{\frac{1}{\lambda_j}} \right] \\
 &= \left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right)^{\frac{2}{\lambda_1} - \sum_{j=2}^m \frac{2}{\lambda_j}} \left\{ \prod_{j=2}^m \left[\left(\sum_{s=1}^n \sum_{r=1}^n A_{s1}^{\lambda_1} A_{rj}^{\lambda_j} (1 - e_r + e_s) \right) \right. \right. \\
 &\quad \cdot \left. \left. \left(\sum_{s=1}^n \sum_{r=1}^n A_{sj}^{\lambda_j} A_{r1}^{\lambda_1} (1 - e_r + e_s) \right) \right]^{\frac{1}{\lambda_j}} \right\} \\
 &= \left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right)^{\frac{2}{\lambda_1} - \sum_{j=2}^m \frac{2}{\lambda_j}} \left\{ \prod_{j=2}^m \left[\left(\sum_{s=1}^n A_{s1}^{\lambda_1} \sum_{r=1}^n A_{rj}^{\lambda_j} \right. \right. \right. \\
 &\quad \left. \left. - \sum_{s=1}^n A_{s1}^{\lambda_1} \sum_{r=1}^n A_{rj}^{\lambda_j} e_r + \sum_{s=1}^n A_{s1}^{\lambda_1} e_s \sum_{r=1}^n A_{rj}^{\lambda_j} \right) \right. \\
 &\quad \cdot \left. \left. \left(\sum_{s=1}^n A_{sj}^{\lambda_j} \sum_{r=1}^n A_{r1}^{\lambda_1} - \sum_{s=1}^n A_{sj}^{\lambda_j} \sum_{r=1}^n A_{r1}^{\lambda_1} e_r + \sum_{s=1}^n A_{sj}^{\lambda_j} e_s \sum_{r=1}^n A_{r1}^{\lambda_1} \right) \right]^{\frac{1}{\lambda_j}} \right\}
 \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right)^{\frac{2}{\lambda_1} - \sum_{j=2}^m \frac{2}{\lambda_j}} \\
 &\quad \cdot \left\{ \prod_{j=2}^m \left[\left(\left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} \right) \right)^2 \right. \right. \\
 &\quad \left. \left. - \left(\left(\sum_{r=1}^n A_{r1}^{\lambda_1} \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} e_r \right) \right. \right. \right. \\
 &\quad \left. \left. \left. - \left(\sum_{r=1}^n A_{r1}^{\lambda_1} e_r \right) \left(\sum_{r=1}^n A_{rj}^{\lambda_j} \right) \right)^2 \right]^{\frac{1}{\lambda_j}} \right\}. \quad (3.26)
 \end{aligned}$$

进而由不等式(3.24)和(3.26)立刻得到我们要证的不等式(3.22). \square

利用上述两个定理, 我们得到一般化的 Hölder 不等式的如下两个改进形式:

推论 3.5 设 $A_{rj} > 0$ ($r = 1, 2, \dots, n$, $j = 1, 2, \dots, m$), $\sum_{j=1}^m \frac{1}{\lambda_j} = 1$, $\sum_{r=1}^n A_{rj}^{\lambda_j} \neq 0$, 并且 $1 - e_r + e_s \geq 0$ ($s = 1, 2, \dots, n$). 如果 $\lambda_1 > 0$, $\lambda_j < 0$ ($j = 2, 3, \dots, m$), 则

$$\begin{aligned}
 \sum_{r=1}^n \prod_{j=1}^m A_{rj} &\geq \left[\prod_{j=1}^m \left(\sum_{r=1}^n A_{rj}^{\lambda_j} \right)^{\frac{1}{\lambda_j}} \right] \\
 &\quad \cdot \left\{ \prod_{j=2}^m \left[1 - \frac{1}{2\lambda_j} \left(\frac{\sum_{r=1}^n A_{r1}^{\lambda_1} e_r}{\sum_{r=1}^n A_{r1}^{\lambda_1}} - \frac{\sum_{r=1}^n A_{rj}^{\lambda_j} e_r}{\sum_{r=1}^n A_{rj}^{\lambda_j}} \right)^2 \right] \right\}. \quad (3.27)
 \end{aligned}$$

推论 3.6 设 $F_j(x)$ 是定义在 $[a, b]$ 上的非负可积函数, 并且 $\int_a^b F_j^{\lambda_j}(x) dx$

存在且不等于 0, 设 $1 - e(x) + e(y) \geq 0$, $\int_a^b e(x) dx < \infty$, 并且

$\sum_{j=1}^m \frac{1}{\lambda_j} = 1$. 如果 $\lambda_1 > 0$, $\lambda_j < 0$ ($j = 2, 3, \dots, m$), 则有

$$\int_a^b \prod_{j=1}^m F_j(x) dx \geq \left[\prod_{j=1}^m \left(\int_a^b F_j^{\lambda_j}(x) dx \right)^{\frac{1}{\lambda_j}} \right] \cdot \left\{ \prod_{j=2}^m \left[1 - \frac{1}{2\lambda_j} \left(\frac{\int_a^b F_1^{\lambda_1}(x) \epsilon(x) dx}{\int_a^b F_1^{\lambda_1}(x) dx} - \frac{\int_a^b F_j^{\lambda_j}(x) \epsilon(x) dx}{\int_a^b F_j^{\lambda_j}(x) dx} \right)^2 \right] \right\}. \quad (3.28)$$

3.4 反向胡克不等式的第三种推广

这一节我们将首先给出胡克得到的胡克不等式的一种反向形式, 然后再给出作者本人得到的该反向形式的推广.

定理 3.7^[14] 设 $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$, $a_k > 0$, $b_k > 0$ ($k = 1, 2, \dots, n$), 并且 $e = (e_1, e_2, \dots, e_n)$, $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$).

如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda = \max \left\{ -1, \frac{1}{q} \right\}$, 则有

$$(\alpha, \beta) \geq (\alpha^p, \epsilon)^{\frac{1}{p}} (\beta^q, \epsilon)^{\frac{1}{q}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, e) - (\alpha, \beta, e)(\beta^q, \epsilon)}{(\alpha, \beta)(\beta^q, \epsilon)} \right)^2 \right]^{\frac{\lambda}{2}}. \quad (3.29)$$

其中, $(\alpha, \beta) = \sum_{k=1}^n a_k b_k$, $\epsilon = (1, 1, \dots, 1)$, $(\alpha^t, \epsilon) = \sum_{k=1}^n a_k^t$, $(\alpha, \beta, e) = \sum_{k=1}^n a_k b_k e_k$.

证 设 $l = \frac{1}{p}$, $l' = -\frac{q}{p}$, 则 $l > 1$, $\frac{1}{l} + \frac{1}{l'} = 1$. 此外设 $\gamma = (a_1^p b_1^p, a_2^p b_2^p, \dots, a_n^p b_n^p)$, $\delta = (b_1^{-p}, b_2^{-p}, \dots, b_n^{-p})$. 在胡克不等式(2.1)中利用以下替换:

$$p \rightarrow l, \quad q \rightarrow l', \quad \alpha \rightarrow \gamma, \quad \beta \rightarrow \delta,$$

有

$$(\gamma, \delta) \leq (\gamma', \varepsilon)^{\frac{1}{l}} (\delta', \varepsilon)^{\frac{1}{l'}} \left[1 - \left(\frac{(\gamma', \varepsilon)(\delta', \varepsilon) - (\gamma', \varepsilon)(\delta', \varepsilon)}{(\gamma', \varepsilon)(\delta', \varepsilon)} \right)^2 \right]^{\frac{1}{2l}}. \quad (3.30)$$

进而

$$(\alpha^p, \varepsilon) \leq (\alpha, \beta)^{\frac{1}{l}} (\beta^q, \varepsilon)^{\frac{1}{l'}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, e)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{\frac{1}{2l}}. \quad (3.31)$$

在不等式(3.31)两端开 p 次方根, 有要证的不等式:

$$(\alpha, \beta) \geq (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, e)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{-\frac{1}{2}}. \quad (3.32)$$

□

定理 3.8^[33] 设 $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$, $a_k > 0$, $b_k > 0$ ($k = 1, 2, \dots, n$), 并且 $e = (e_1, e_2, \dots, e_n)$, $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$).

如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} \geq 0$, $\rho = \max \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}$, $\lambda = \max \left\{ -1, \frac{1}{q} \right\}$, 则有

$$(\alpha, \beta) \geq n^{1-\rho} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, e)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{\frac{\lambda}{2}}. \quad (3.33)$$

其中, $(\alpha, \beta) = \sum_{k=1}^n a_k b_k$, $\varepsilon = (1, 1, \dots, 1)$, $(\alpha^t, \varepsilon) = \sum_{k=1}^n a_k^t$, $(\alpha, \beta, e) = \sum_{k=1}^n a_k b_k e_k$.

证 (1) 当 $-1 \leq q < 0$ 时, 因为 $\frac{1}{p} + \frac{1}{q} \geq 0$, 于是 $0 < p \leq 1$. 设

$l = \frac{1}{p}$, $l' = -\frac{q}{p}$, 则 $l \geq l' > 0$. 此外, 设 $\gamma = (a_1^p b_1^p, a_2^p b_2^p, \dots, a_n^p b_n^p)$, $\delta =$

$(b_1^{-p}, b_2^{-p}, \dots, b_n^{-p})$. 当 $0 < \frac{1}{l} + \frac{1}{l'} < 1$ 时, 在(2.8)中利用以下替换:

$$p \rightarrow l, q \rightarrow l', \alpha \rightarrow \gamma, \beta \rightarrow \delta,$$

有

$$(\gamma, \delta) \leq n^{1-\frac{1}{l}-\frac{1}{l'}} (\gamma^l, \varepsilon)^{\frac{1}{l}} (\delta^{l'}, \varepsilon)^{\frac{1}{l'}} \left[1 - \left(\frac{(\gamma^l, \varepsilon)(\delta^{l'}, \varepsilon) - (\gamma^l, \varepsilon)(\delta^{l'}, \varepsilon)}{(\gamma^l, \varepsilon)(\delta^{l'}, \varepsilon)} \right)^2 \right]^{\frac{1}{2l}}. \quad (3.34)$$

进而

$$(\alpha^p, \varepsilon) \leq n^{1-\frac{1}{p}-\frac{1}{p'}} (\alpha, \beta)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{p'}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, \varepsilon)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{\frac{1}{2l}}. \quad (3.35)$$

现在在不等式(3.35)两端开 p 次方根, 有

$$(\alpha, \beta) \geq n^{1-\frac{1}{p}-\frac{1}{q}} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, \varepsilon)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{-\frac{1}{2}}. \quad (3.36)$$

由于 $0 < \frac{1}{l} + \frac{1}{l'} < 1$, 从而 $\frac{1}{p} + \frac{1}{q} > 1$. 于是

$$\begin{aligned} (\alpha, \beta) &\geq n^{1-\max\{\frac{1}{p}+\frac{1}{q}, 1\}} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \\ &\quad \cdot \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, \varepsilon)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{-\frac{1}{2}}. \end{aligned} \quad (3.37)$$

当 $\frac{1}{l} + \frac{1}{l'} \geq 1$ 时, 有 $\frac{1}{p} + \frac{1}{q} \leq 1$. 类似地, 有

$$\begin{aligned} (\alpha, \beta) &\geq (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \\ &\quad \cdot \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, \varepsilon)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{-\frac{1}{2}} \\ &= n^{1-\max\{\frac{1}{p}+\frac{1}{q}, 1\}} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \\ &\quad \cdot \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, \varepsilon) - (\alpha, \beta, \varepsilon)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{-\frac{1}{2}}. \end{aligned} \quad (3.38)$$

(2) 当 $q < -1$ 时, 采用与(1)类似的方法可得

$$(\alpha, \beta) \geq n^{1-\max\left\{\frac{1}{p}+\frac{1}{q}, 1\right\}} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \cdot \left[1 - \left(\frac{(\alpha, \beta)(\beta^q, e) - (\alpha, \beta, e)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]^{\frac{1}{2q}}. \quad (3.39)$$

于是得到要证的不等式(3.33). \square

利用定理 3.8, 易得 Hölder 不等式的如下推广和改进:

推论 3.7^[33] 设 $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$, $a_k > 0$, $b_k > 0$ ($k = 1, 2, \dots, n$), 并且 $e = (e_1, e_2, \dots, e_n)$, $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$).

如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} \geq 0$, $\rho = \max\left\{\frac{1}{p} + \frac{1}{q}, 1\right\}$, $\lambda = \max\left\{-1, \frac{1}{q}\right\}$, 则有

$$(\alpha, \beta) \geq n^{1-\rho} (\alpha^p, \varepsilon)^{\frac{1}{p}} (\beta^q, \varepsilon)^{\frac{1}{q}} \left[1 - \frac{\lambda}{2} \left(\frac{(\alpha, \beta)(\beta^q, e) - (\alpha, \beta, e)(\beta^q, \varepsilon)}{(\alpha, \beta)(\beta^q, \varepsilon)} \right)^2 \right]. \quad (3.40)$$

特别地, 在(3.40)中, 如果取 $\frac{1}{p} + \frac{1}{q} = 1$, 则由上述推论可得 Hölder 不等式的如下改进:

推论 3.8^[33] 设 $a_k > 0$, $b_k > 0$ ($k = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 并且 $\lambda = \max\left\{-1, \frac{1}{q}\right\}$, 则有

$$\sum_{k=1}^n a_k b_k \geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n b_k^q \right)^{\frac{1}{q}} \left[1 - \frac{\lambda}{2} \left(\frac{\sum_{k=1}^n b_k^q e_k}{\sum_{k=1}^n b_k^q} - \frac{\sum_{k=1}^n a_k b_k e_k}{\sum_{k=1}^n a_k b_k} \right)^2 \right]. \quad (3.41)$$

定理 3.9^[33] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x) > 0$, $g(x) > 0$, $1 - e(x) + e(y) \geq 0$. 如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} \geq 1$, $\lambda = \max\left\{-1, \frac{1}{q}\right\}$, 则有

$$\int_a^b f(x)g(x)dx \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} \\ \cdot \left[1 - \left(\frac{\int_a^b f(x)g(x)e(x)dx}{\int_a^b f(x)g(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right]^{\frac{\lambda}{2}}. \quad (3.42)$$

证 对于任意的正整数 n , 将区间 $[a, b]$ n 等分:

$$a < a + \frac{b-a}{n} < \cdots < a + \frac{b-a}{n}k < \cdots < a + \frac{b-a}{n}(n-1) < b,$$

$$x_k = a + \frac{b-a}{n}k, \quad \Delta x_k = \frac{b-a}{n}, \quad k = 1, 2, \cdots, n.$$

由定理 3.8, 我们得到以下不等式:

$$\sum_{k=1}^n f(x_k)g(x_k) \geq n^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k) \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(x_k) \right)^{\frac{1}{q}} \\ \cdot \left[1 - \left(\frac{\sum_{k=1}^n f(x_k)g(x_k)e(x_k)}{\sum_{k=1}^n f(x_k)g(x_k)} - \frac{\sum_{k=1}^n g^q(x_k)e(x_k)}{\sum_{k=1}^n g^q(x_k)} \right)^2 \right]^{\frac{\lambda}{2}}. \quad (3.43)$$

也就是

$$\sum_{k=1}^n f(x_k)g(x_k) \frac{b-a}{n} \\ \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\sum_{k=1}^n f^p(x_k) \frac{b-a}{n} \right)^{\frac{1}{p}} \left(\sum_{k=1}^n g^q(x_k) \frac{b-a}{n} \right)^{\frac{1}{q}} \\ \cdot \left[1 - \left(\frac{\sum_{k=1}^n f(x_k)g(x_k)e(x_k) \frac{b-a}{n}}{\sum_{k=1}^n f(x_k)g(x_k) \frac{b-a}{n}} - \frac{\sum_{k=1}^n g^q(x_k)e(x_k) \frac{b-a}{n}}{\sum_{k=1}^n g^q(x_k) \frac{b-a}{n}} \right)^2 \right]^{\frac{\lambda}{2}}. \quad (3.44)$$

因为 $f(x), g(x), e(x)$ 是正的黎曼可积函数, 于是 $f^p(x), g^q(x), g^q(x)e(x)$ 在 $[a, b]$ 上也是可积的. 在不等式(3.44)两端令 $n \rightarrow \infty$, 于是得到要证的不等式(3.42). \square

利用定理 3.9, 易得积分型 Hölder 不等式的如下推广和改进:

推论 3.9^[33] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x) > 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} \geq 1, \lambda =$

$\max \left\{ -1, \frac{1}{q} \right\}$, 则有

$$\int_a^b f(x)g(x)dx \geq (b-a)^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} \cdot \left[1 - \frac{\lambda}{2} \left(\frac{\int_a^b f(x)g(x)e(x)dx}{\int_a^b f(x)g(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right], \quad (3.45)$$

例 3.1^[33] 在(3.45)中, 取 $a = 0, e(x) = \frac{1}{2} \cos \frac{2\pi x}{b}$, 则有

$$\int_0^b f(x)g(x)dx \geq b^{1-\frac{1}{p}-\frac{1}{q}} \left(\int_0^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^b g^q(x)dx \right)^{\frac{1}{q}} \cdot \left[1 - \frac{\lambda}{8} \left(\frac{\int_0^b f(x)g(x) \cos \frac{2\pi x}{b} dx}{\int_0^b f(x)g(x)dx} - \frac{\int_0^b g^q(x) \cos \frac{2\pi x}{b} dx}{\int_0^b g^q(x)dx} \right)^2 \right], \quad (3.46)$$

其中 $\lambda = \max \left\{ -1, \frac{1}{q} \right\}$.

特别地, 在(3.45)中取 $\frac{1}{p} + \frac{1}{q} = 1$, 由推论 3.9, 我们得到以下改进的 Hölder 不等式:

推论 3.10^[33] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x) > 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\int_a^b f(x)g(x)dx \geq \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} \cdot \left[1 - \frac{\lambda}{2} \left(\frac{\int_a^b f(x)g(x)e(x)dx}{\int_a^b f(x)g(x)dx} - \frac{\int_a^b g^q(x)e(x)dx}{\int_a^b g^q(x)dx} \right)^2 \right], \quad (3.47)$$

其中 $\lambda = \max \left\{ -1, \frac{1}{q} \right\}$.

采用和定理 3.8 类似的方法, 我们得到胡克不等式的如下反向形式:

定理 3.10^[33] 设 E 是可测集, $f(x), g(x)$ 是 E 上正的可测函数, 并且

$\int_E f^p(x)dx < \infty, \int_E g^q(x)dx < \infty, e(x)$ 是可测函数, $\int_E e(x)dx < \infty$,

$1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\int_E f(x)g(x)dx \geq \left(\int_E f^p(x)dx \right)^{\frac{1}{p}} \left(\int_E g^q(x)dx \right)^{\frac{1}{q}} \cdot \left[1 - \left(\frac{\int_E f(x)g(x)e(x)dx}{\int_E f(x)g(x)dx} - \frac{\int_E g^q(x)e(x)dx}{\int_E g^q(x)dx} \right)^2 \right]^{\frac{\lambda}{2}}, \quad (3.48)$$

其中 $\lambda = \max \left\{ -1, \frac{1}{q} \right\}$.

第4章

几个重要不等式构成的函数的单调性性质

在这一章中, 我们介绍胡克不等式、反向胡克不等式、Hölder 不等式及 Minkowski 不等式构成的函数的单调性性质.

4.1 胡克不等式构成的函数的单调性

定理 4.1^[13] 设 $A_r, B_r \geq 0$, $1 - e_r + e_k \geq 0$ ($r, k = 1, 2, \dots$), 以及 $p \geq q > 0$,

$\frac{1}{p} + \frac{1}{q} = 1$. 记

$$F(n) = \left(\sum_{r=1}^n A_r B_r \right)^2 - \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (4.1)$$

则有

$$F(n) \geq F(n+1). \quad (4.2)$$

这个定理的积分形式如下:

定理 4.2^[13] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x), g(x) \geq 0, 1 - e(x) + e(y) \geq 0$. 如果 $p \geq q > 0, \frac{1}{p} + \frac{1}{q} = 1$, 并且记

$$\begin{aligned} G(t) = & \left(\int_a^t f(x)g(x)dx \right)^2 - \left(\int_a^t f^p(x)dx \right)^{\frac{2}{p} - \frac{2}{q}} \\ & \cdot \left[\left(\int_a^t f^p(x)dx \int_a^t g^q(x)dx \right)^2 - \left(\int_a^t f^p(x)e(x)dx \int_a^t g^q(x)dx \right. \right. \\ & \left. \left. - \int_a^t f^p(x)dx \int_a^t g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{q}}. \end{aligned} \quad (4.3)$$

则有

$$G(t_1) \geq G(t_2), \quad a \leq t_1 \leq t_2 \leq b. \quad (4.4)$$

证明可参考文献 [13].

注意: 对于(4.1), 如果我们令 $n = 1$, 则有 $F(1) = 0$. 进而由定理 4.1 就可以得到离散型的胡克不等式. 类似地, 对于(4.4), 如果设 $t_1 = a$, $t_2 = b$, 利用定理 4.2 就可以得到积分型的胡克不等式.

4.2 反向胡克不等式构成的函数的单调性

定理 4.3^[35] 设 $A_r \geq 0, B_r > 0, 1 - e_r + e_k \geq 0 (r, k = 1, 2, \dots)$ 及 $q < 0$,

$\frac{1}{p} + \frac{1}{q} = 1$. 记

$$\begin{aligned} F(n) = & \left(\sum_{r=1}^n A_r B_r \right)^2 - \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\ & \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q e_r \right) \right]^2 \right\}^{\frac{1}{q}}. \end{aligned} \quad (4.5)$$

则有

$$F(n) \leq F(n+1). \quad (4.6)$$

这个定理的积分形式如下:

定理 4.4^[35] 设 $f(x), g(x), e(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且

$f(x) \geq 0, g(x) > 0, 1 - e(x) + e(y) \geq 0$. 如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 并且

记

$$G(t) = \left(\int_a^t f(x)g(x)dx \right)^2 - \left(\int_a^t f^p(x)dx \right)^{\frac{2}{p}-\frac{2}{q}} \left[\left(\int_a^t f^p(x)dx \int_a^t g^q(x)dx \right)^2 - \left(\int_a^t f^p(x)e(x)dx \int_a^t g^q(x)dx - \int_a^t f^p(x)dx \int_a^t g^q(x)e(x)dx \right)^2 \right]^{\frac{1}{q}}, \quad (4.7)$$

则有

$$G(t_1) \leq G(t_2), \quad a \leq t_1 \leq t_2 \leq b. \quad (4.8)$$

证 这里仅仅给出不等式(4.6)的证明. 不等式(4.8)的证明可以类似地给出. 经过一些简单的运算, 有

$$\begin{aligned} & \sum_{r=1}^N A_r B_r \sum_{k=1}^N A_k B_k (1 - e_r + e_k) \\ &= \sum_{r=1}^N \sum_{k=1}^N A_r B_r A_k B_k - \sum_{r=1}^N \sum_{k=1}^N A_r B_r A_k B_k e_r + \sum_{r=1}^N \sum_{k=1}^N A_r B_r A_k B_k e_k \\ &= \left(\sum_{r=1}^N A_r B_r \right)^2. \end{aligned} \quad (4.9)$$

对于任意的 $C_{kr} > 0, D_{kr} > 0, k, r = 1, 2, \dots$, 令

$$X_{nr} = \sum_{k=1}^n C_{kr}^p, \quad Y_{nr} = \sum_{k=1}^n D_{kr}^q.$$

一方面, 对于 $t, s > 0$, 由定理 1.10 可知

$$\begin{aligned} A_r B_r X_{nr}^{\frac{1}{p}} Y_{nr}^{\frac{1}{q}} &= t B_r X_{nr}^{\frac{1}{q}} t^{-1} \left(A_r^{\frac{p}{q}} Y_{nr}^{\frac{1}{q}} A_r^{1-\frac{p}{q}} X_{nr}^{\frac{1}{p}-\frac{1}{q}} \right) \\ &\geq \frac{1}{q} t^q B_r^q X_{nr} + \frac{1}{p} t^{-p} \left(A_r^{\frac{p}{q}} Y_{nr}^{\frac{1}{q}} A_r^{1-\frac{p}{q}} X_{nr}^{\frac{1}{p}-\frac{1}{q}} \right)^p \\ &= \frac{1}{q} t^q B_r^q X_{nr} + \frac{1}{p} t^{-p} s A_r^{\frac{p^2}{q}} Y_{nr}^{\frac{p}{q}} s^{-1} A_r^{p-\frac{p^2}{q}} X_{nr}^{1-\frac{p}{q}} \end{aligned}$$

$$\begin{aligned}
 &\geq \frac{1}{q} t^q B_r^q X_{nr} + \frac{1}{p} t^{-p} \left[\frac{p}{q} s^{\frac{q}{p}} A_r^p Y_{nr} + \left(1 - \frac{p}{q}\right) s^{\frac{q}{p-q}} A_r^p X_{nr} \right] \\
 &= \frac{1}{q} t^q B_r^q X_{nr} + \frac{1}{q} t^{-p} s^{\frac{q}{p}} A_r^p Y_{nr} + \left(\frac{1}{p} - \frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} A_r^p X_{nr}. \quad (4.10)
 \end{aligned}$$

另一方面, 由 Hölder 不等式有

$$\sum_{k=1}^n C_{kr} D_{kr} \geq \left(\sum_{k=1}^n C_{kr}^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n D_{kr}^q \right)^{\frac{1}{q}} = X_{nr}^{\frac{1}{p}} Y_{nr}^{\frac{1}{q}}. \quad (4.11)$$

进而, 对于任意的 $t > 0, s > 0$, 如果设

$$\begin{aligned}
 F(m, n; s, t) &= \sum_{r=1}^m A_r B_r \left(\sum_{k=1}^n C_{kr} D_{kr} \right) - \frac{1}{q} t^q \sum_{r=1}^m B_r^q X_{nr} \\
 &\quad - \frac{1}{q} t^{-p} s^{\frac{q}{p}} \sum_{r=1}^m A_r^p Y_{nr} - \left(\frac{1}{p} - \frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} \sum_{r=1}^m A_r^p X_{nr}, \quad (4.12)
 \end{aligned}$$

则利用不等式(4.11)和(4.10), 得到

$$\begin{aligned}
 &F(m+1, n; s, t) - F(m, n; s, t) \\
 &= A_{m+1} B_{m+1} \sum_{k=1}^n C_{k(m+1)} D_{k(m+1)} - \frac{1}{q} t^q B_{m+1}^q X_{n(m+1)} \\
 &\quad - \frac{1}{q} t^{-p} s^{\frac{q}{p}} A_{m+1}^p Y_{n(m+1)} - \left(\frac{1}{p} - \frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} A_{m+1}^p X_{n(m+1)} \\
 &\geq A_{m+1} B_{m+1} X_{n(m+1)}^{\frac{1}{p}} Y_{n(m+1)}^{\frac{1}{q}} - \frac{1}{q} t^q B_{m+1}^q X_{n(m+1)} \\
 &\quad - \frac{1}{q} t^{-p} s^{\frac{q}{p}} A_{m+1}^p Y_{n(m+1)} - \left(\frac{1}{p} - \frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} A_{m+1}^p X_{n(m+1)} \\
 &\geq 0 \quad (4.13)
 \end{aligned}$$

和

$$\begin{aligned}
 &F(m, n+1; s, t) - F(m, n; s, t) \\
 &= \sum_{r=1}^m A_r B_r C_{(n+1)r} D_{(n+1)r} - \frac{1}{q} t^q \sum_{r=1}^m B_r^q C_{(n+1)r}^p \\
 &\quad - \frac{1}{q} t^{-p} s^{\frac{q}{p}} \sum_{r=1}^m A_r^p D_{(n+1)r}^q - \left(\frac{1}{p} - \frac{1}{q}\right) t^{-p} s^{\frac{q}{p-q}} \sum_{r=1}^m A_r^p C_{(n+1)r}^p \\
 &\geq 0. \quad (4.14)
 \end{aligned}$$

进而, 由不等式(4.13)和(4.14)有

$$F(n, n; s, t) \leq F(n+1, n; s, t) \leq F(n+1, n+1; s, t). \quad (4.15)$$

此外考虑到

$$\frac{\partial F(n, n; s, t)}{\partial s} = -\frac{1}{p} t^{-p} s^{\frac{q-p}{p}} \sum_{r=1}^n A_r^p Y_{nr} + \frac{1}{p} t^{-p} s^{\frac{2q-p}{p-q}} \sum_{r=1}^n A_r^p X_{nr}, \quad (4.16)$$

$$\begin{aligned} \frac{\partial F(n, n; s, t)}{\partial t} &= -t^{q-1} \sum_{r=1}^n B_r^q X_{nr} + \frac{p}{q} t^{-p-1} s^{\frac{q}{p}} \sum_{r=1}^n A_r^p Y_{nr} \\ &\quad + \left(1 - \frac{p}{q}\right) t^{-p-1} s^{\frac{q}{p-q}} \sum_{r=1}^n A_r^p X_{nr}, \end{aligned} \quad (4.17)$$

由下面两个方程

$$\frac{\partial F(n, n; s, t)}{\partial s} = 0, \quad \frac{\partial F(n, n; s, t)}{\partial t} = 0,$$

我们得到

$$\begin{aligned} s_0 &= \left(\frac{\sum_{r=1}^n A_r^p X_{nr}}{\sum_{r=1}^n A_r^p Y_{nr}} \right)^{\frac{p(q-p)}{q^2}}, \\ t_0 &= \frac{1}{\left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{pq}}} \left[\left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{1}{q^2}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{pq} - \frac{1}{q^2}} \right]. \end{aligned}$$

从而, 由不等式(4.16)和(4.17)有

$$\begin{aligned} \frac{\partial^2 F(n, n; s, t)}{\partial s^2} &= \frac{p-q}{p^2} t^{-p} s^{\frac{q-2p}{p}} \sum_{r=1}^n A_r^p Y_{nr} \\ &\quad + \frac{2q-p}{p(p-q)} t^{-p} s^{\frac{3q-2p}{p-q}} \sum_{r=1}^n A_r^p X_{nr}, \end{aligned} \quad (4.18)$$

$$\frac{\partial^2 F(n, n; s, t)}{\partial s \partial t} = t^{-p-1} s^{\frac{q-p}{p}} \sum_{r=1}^n A_r^p Y_{nr} - t^{-p-1} s^{\frac{2q-p}{p-q}} \sum_{r=1}^n A_r^p X_{nr}, \quad (4.19)$$

$$\begin{aligned} \frac{\partial^2 F(n, n; s, t)}{\partial t^2} = & (1-q)t^{q-2} \sum_{r=1}^n B_r^q X_{nr} - \frac{p(p+1)}{q} t^{-p-2} s^{\frac{q}{p}} \sum_{r=1}^n A_r^p Y_{nr} \\ & + \frac{(p-q)(p+1)}{q} t^{-p-2} s^{\frac{q}{p-q}} \sum_{r=1}^n A_r^p X_{nr}. \end{aligned} \quad (4.20)$$

经过一些简单的运算, 得到

$$\begin{aligned} \left. \frac{\partial^2 F(n, n; s, t)}{\partial s^2} \right|_{(s_0, t_0)} &= \frac{q^2}{p^2(p-q)} \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{3pq-2p^2-p}{q^2}} \\ &\cdot \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{q^2-3pq+2p^2+p-q}{q^2}} > 0. \end{aligned} \quad (4.21)$$

$$\left. \frac{\partial^2 F(n, n; s, t)}{\partial s \partial t} \right|_{(s_0, t_0)} = 0, \quad (4.22)$$

$$\begin{aligned} \left. \frac{\partial^2 F(n, n; s, t)}{\partial t^2} \right|_{(s_0, t_0)} &= -pq \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{p+2}{pq}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{q-2}{q^2}} \\ &\cdot \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{q^2-pq-2q+2p}{pq^2}} > 0. \end{aligned} \quad (4.23)$$

如果设

$$A = \left. \frac{\partial^2 F(n, n; s, t)}{\partial s^2} \right|_{(s_0, t_0)},$$

$$B = \left. \frac{\partial^2 F(n, n; s, t)}{\partial s \partial t} \right|_{(s_0, t_0)},$$

$$C = \left. \frac{\partial^2 F(n, n; s, t)}{\partial t^2} \right|_{(s_0, t_0)},$$

则由(4.21), (4.22)和(4.23)有

$$AC - B^2 > 0. \quad (4.24)$$

进而, 由上面的不等式(4.24)和(4.21)得到

$$\min_{t, s > 0} \{F(n, n; s, t)\} = F(n, n; s_0, t_0), \quad (4.25)$$

从而

$$\begin{aligned}
 & \min_{t,s>0} \{F(n, n; s, t)\} \\
 &= F(n, n; s_0, t_0) \\
 &= \sum_{r=1}^n A_r B_r \left(\sum_{k=1}^n C_{kr} D_{kr} \right) \\
 &= \frac{\left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n B_r^q X_{nr} \right)}{q \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{p}}} \\
 &= \frac{\left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{1-\frac{p}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)}{q \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{p}{q^2}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{1-\frac{p}{q}}} \\
 &= \frac{\left(\frac{1}{p} - \frac{1}{q} \right) \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{p}{q}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)}{\left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{p}{q^2}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{p}{q}}} \\
 &= \sum_{r=1}^n A_r B_r \left(\sum_{k=1}^n C_{kr} D_{kr} \right) \\
 &\quad - \frac{1}{q} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}} \\
 &\quad - \frac{1}{p} \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{p}{q}-\frac{p}{q^2}} \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{1-\frac{1}{q}-\frac{p}{q}+\frac{p}{q^2}} \\
 &= \sum_{r=1}^n A_r B_r \left(\sum_{k=1}^n C_{kr} D_{kr} \right) \\
 &\quad - \left(\sum_{r=1}^n A_r^p X_{nr} \right)^{\frac{1}{p}-\frac{1}{q}} \left(\sum_{r=1}^n A_r^p Y_{nr} \right)^{\frac{1}{q}} \left(\sum_{r=1}^n B_r^q X_{nr} \right)^{\frac{1}{q}}.
 \end{aligned}
 \tag{4.26}$$

类似地, 如果设

$$s_1 = \left(\frac{\sum_{r=1}^{n+1} A_r^p X_{(n+1)r}}{\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r}} \right)^{\frac{p(q-p)}{q^2}},$$

$$t_1 = \frac{1}{\left(\sum_{r=1}^{n+1} B_r^q X_{(n+1)r} \right)^{\frac{1}{pq}}} \left[\left(\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r} \right)^{\frac{1}{q^2}} \left(\sum_{r=1}^{n+1} A_r^p X_{(n+1)r} \right)^{\frac{1}{pq} - \frac{1}{q^2}} \right].$$

则

$$\begin{aligned} & F(n+1, n+1; s_1, t_1) \\ &= \sum_{r=1}^{n+1} A_r B_r \left(\sum_{k=1}^{n+1} C_{kr} D_{kr} \right) \\ & \quad - \left(\sum_{r=1}^{n+1} A_r^p X_{(n+1)r} \right)^{\frac{1}{p} - \frac{1}{q}} \left(\sum_{r=1}^{n+1} A_r^p Y_{(n+1)r} \right)^{\frac{1}{q}} \left(\sum_{r=1}^{n+1} B_r^q X_{(n+1)r} \right)^{\frac{1}{q}}. \end{aligned} \quad (4.27)$$

由(4.15), 有

$$F(n, n; s_1, t_1) \leq F(n+1, n+1; s_1, t_1). \quad (4.28)$$

进而由(4.25)和(4.28), 得到

$$F(n, n; s_0, t_0) \leq F(n+1, n+1; s_1, t_1). \quad (4.29)$$

设

$$C_{kr} = A_k(1 - e_r + e_k)^{\frac{1}{p}}, \quad D_{kr} = B_k(1 - e_r + e_k)^{\frac{1}{q}}.$$

于是

$$\begin{aligned} & F(n, n; s_0, t_0) \\ &= \sum_{r=1}^n A_r B_r \sum_{k=1}^n A_k B_k (1 - e_r + e_k) \\ & \quad - \left[\sum_{r=1}^n A_r^p \sum_{k=1}^n B_k^q (1 - e_r + e_k) \right]^{\frac{1}{p} - \frac{1}{q}} \\ & \quad \cdot \left[\sum_{r=1}^n A_r^p \sum_{k=1}^n B_k^q (1 - e_r + e_k) \right]^{\frac{1}{q}} \left[\sum_{r=1}^n B_r^q \sum_{k=1}^n A_k^p (1 - e_r + e_k) \right]^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
 &= \left(\sum_{r=1}^n A_r B_r \right)^2 - \left[\left(\sum_{r=1}^n A_r^p \right)^2 \right]^{\frac{1}{p} - \frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n A_r^p \sum_{k=1}^n B_k^q - \sum_{r=1}^n A_r^p e_r \sum_{k=1}^n B_k^q + \sum_{r=1}^n A_r^p \sum_{k=1}^n B_k^q e_k \right)^{\frac{1}{q}} \\
 &\quad \cdot \left(\sum_{r=1}^n B_r^q \sum_{k=1}^n A_k^p - \sum_{r=1}^n B_r^q e_r \sum_{k=1}^n A_k^p + \sum_{r=1}^n B_r^q \sum_{k=1}^n A_k^p e_k \right)^{\frac{1}{q}} \\
 &= \left(\sum_{r=1}^n A_r B_r \right)^2 - \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^n A_r^p \right) \left(\sum_{r=1}^n B_r^q \right) \right]^2 \right. \\
 &\quad \left. - \left[\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) - \left(\sum_{r=1}^n B_r^q e_r \right) \left(\sum_{r=1}^n A_r^p \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (4.30)
 \end{aligned}$$

类似地, 有

$$\begin{aligned}
 &F(n+1, n+1; s_1, t_1) \\
 &= \left(\sum_{r=1}^{n+1} A_r B_r \right)^2 - \left(\sum_{r=1}^{n+1} A_r^p \right)^{\frac{2}{p} - \frac{2}{q}} \left\{ \left[\left(\sum_{r=1}^{n+1} A_r^p \right) \left(\sum_{r=1}^{n+1} B_r^q \right) \right]^2 \right. \\
 &\quad \left. - \left[\left(\sum_{r=1}^{n+1} A_r^p e_r \right) \left(\sum_{r=1}^{n+1} B_r^q \right) - \left(\sum_{r=1}^{n+1} B_r^q e_r \right) \left(\sum_{r=1}^{n+1} A_r^p \right) \right]^2 \right\}^{\frac{1}{q}}. \quad (4.31)
 \end{aligned}$$

由(4.29), (4.30)和(4.31)即可得到我们要证的不等式(4.6). \square

注意: 对于(4.5)如果令 $n = 1$, 则有 $F(1) = 0$. 进而由定理 4.3 就可以得到离散型的反向胡克不等式. 类似地, 对于(4.8)如果设 $t_1 = a$, $t_2 = b$, 利用定理 4.4 就可以得到积分型的反向胡克不等式.

4.3 Hölder 不等式构成的函数的单调性

在(4.1)和(4.5)中分别取

$$\left(\sum_{r=1}^n A_r^p e_r \right) \left(\sum_{r=1}^n B_r^q \right) = \left(\sum_{r=1}^n B_r^q e_r \right) \left(\sum_{r=1}^n A_r^p \right),$$

则由定理 4.1 和定理 4.3 可得 Hölder 不等式构成函数的如下单调性性质:

定理 4.5^{[35][13]} 设 $A_r \geq 0, B_r > 0$ ($r = 1, 2, \dots$), 并且设

$$F(n) = \left(\sum_{r=1}^n A_r B_r \right)^2 - \left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{2}{q}}. \quad (4.32)$$

如果 $p \geq q > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$F(n) \geq F(n+1) \geq 0; \quad (4.33)$$

如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$0 \leq F(n) \leq F(n+1). \quad (4.34)$$

类似地, 在(4.3)和(4.7)中分别取

$$\int_a^t f^p(x) e(x) dx \int_a^t g^q(x) dx \equiv \int_a^t g^q(x) e(x) dx \int_a^t f^p(x) dx,$$

则由定理 4.2 和定理 4.4 可得积分型 Hölder 不等式构成函数的如下单调性性质:

定理 4.6^{[35][13]} 设 $f(x), g(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) \geq 0, f(x) \geq 0, g(x) > 0$, 设

$$G(t) = \left(\int_a^t f(x) g(x) dx \right)^2 - \left(\int_a^t f^p(x) dx \right)^{\frac{2}{p}} \left(\int_a^t g^q(x) dx \right)^{\frac{2}{q}}. \quad (4.35)$$

如果 $p \geq q > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$G(t_1) \geq G(t_2) \geq 0, \quad a \leq t_1 \leq t_2 \leq b, \quad (4.36)$$

如果 $q < 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$0 \leq G(t_1) \leq G(t_2), \quad a \leq t_1 \leq t_2 \leq b. \quad (4.37)$$

由定理 4.5 和定理 4.6 可得如下 Hölder 不等式的改进:

推论 4.1 设 $A_r \geq 0, B_r > 0$ ($r = 1, 2, \dots, n$). 如果 $p \geq q > 0, \frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\sum_{r=1}^n A_r B_r \leq \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} (1 + \rho)^{\frac{1}{2}}, \quad (4.38)$$

其中

$$\rho = \frac{(A_1 B_1 + A_2 B_2)^2 - (A_1^p + A_2^p)^{\frac{2}{p}} (B_1^q + B_2^q)^{\frac{2}{q}}}{\left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{2}{q}}} \leq 0;$$

如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\sum_{r=1}^n A_r B_r \geq \left(\sum_{r=1}^n A_r^p \right)^{\frac{1}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{1}{q}} (1 + \pi)^{\frac{1}{2}}, \quad (4.39)$$

其中

$$\pi = \frac{(A_1 B_1 + A_2 B_2)^2 - (A_1^p + A_2^p)^{\frac{2}{p}} (B_1^q + B_2^q)^{\frac{2}{q}}}{\left(\sum_{r=1}^n A_r^p \right)^{\frac{2}{p}} \left(\sum_{r=1}^n B_r^q \right)^{\frac{2}{q}}} \geq 0.$$

推论 4.2 设 $f(x), g(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) \geq 0$,

$g(x) > 0$. 如果 $p \geq q > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\int_a^b f(x)g(x)dx \leq \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} (1 + \theta)^{\frac{1}{2}}, \quad (4.40)$$

其中

$$\theta = \frac{\left(\int_a^{\frac{a+b}{2}} f(x)g(x)dx \right)^2 - \left(\int_a^{\frac{a+b}{2}} f^p(x)dx \right)^{\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} g^q(x)dx \right)^{\frac{2}{q}}}{\left(\int_a^b f^p(x)dx \right)^{\frac{2}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{2}{q}}}$$

≤ 0 ;

如果 $q < 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 则有

$$\int_a^b f(x)g(x)dx \geq \left(\int_a^b f^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^b g^q(x)dx \right)^{\frac{1}{q}} (1 + \eta)^{\frac{1}{2}}, \quad (4.41)$$

其中

$$\eta = \frac{\left(\int_a^{\frac{a+b}{2}} f(x)g(x)dx\right)^2 - \left(\int_a^{\frac{a+b}{2}} f^p(x)dx\right)^{\frac{2}{p}} \left(\int_a^{\frac{a+b}{2}} g^q(x)dx\right)^{\frac{2}{q}}}{\left(\int_a^b f^p(x)dx\right)^{\frac{2}{p}} \left(\int_a^b g^q(x)dx\right)^{\frac{2}{q}}} \geq 0.$$

4.4 Minkowski 不等式构成的函数的单调性

定理4.7^{[35][13]} 设 $f(x), g(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) > 0$, $g(x) > 0$. 设

$$\begin{aligned} M(t) = & \left[\int_a^t (f(x) + g(x))^{p-1} f(x) dx \right]^2 + \left[\int_a^t (f(x) + g(x))^{p-1} g(x) dx \right]^2 \\ & - \left[\left(\int_a^t f^p(x) dx \right)^{\frac{2}{p}} + \left(\int_a^t g^p(x) dx \right)^{\frac{2}{p}} \right] \left[\int_a^t (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}}. \end{aligned} \quad (4.42)$$

如果 $0 < p < 1$, 则有

$$0 \leq M(t_1) \leq M(t_2), \quad a \leq t_1 \leq t_2 \leq b; \quad (4.43)$$

如果 $p > 1$, 则有

$$M(t_1) \geq M(t_2) \geq 0, \quad a \leq t_1 \leq t_2 \leq b. \quad (4.44)$$

证 这里只证明 $0 < p < 1$ 时的情形, $p > 1$ 时的情形可以类似地给出. 记

$$\begin{aligned} M_f(t) = & \left[\int_a^t (f(x) + g(x))^{p-1} f(x) dx \right]^2 \\ & - \left[\int_a^t f^p(x) dx \right]^{\frac{2}{p}} \left[\int_a^t (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}}, \\ M_g(t) = & \left[\int_a^t (f(x) + g(x))^{p-1} g(x) dx \right]^2 \\ & - \left[\int_a^t g^p(x) dx \right]^{\frac{2}{p}} \left[\int_a^t (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}}. \end{aligned}$$

由定理 4.6 可知函数 $M_f(t)$ 和 $M_g(t)$ 在区间 $[a, b]$ 上关于 t 是递增的. 又由于 $M(t) = M_f(t) + M_g(t)$, 故有

$$0 \leq M(t_1) \leq M(t_2), \quad a \leq t_1 \leq t_2 \leq b. \quad \square$$

评注 4.1 一方面, 由定理 4.7 可知在 $0 < p < 1$ 时, $M(b) \geq 0$. 另一方面, 由 Hölder 不等式可知

$$\begin{aligned} & \left[\int_a^b (f(x) + g(x))^{p-1} f(x) dx \right] \left[\int_a^b (f(x) + g(x))^{p-1} g(x) dx \right] \\ & \geq \left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^b g^p(x) dx \right)^{\frac{1}{p}} \left[\int_a^b (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}}. \end{aligned} \quad (4.45)$$

进而由上述不等式及 $M(b) \geq 0$ 可得 Minkowski 不等式在 $0 < p < 1$ 时的结论.

类似地, 可得 Minkowski 不等式在 $p \geq 1$ 时的结论.

由定理 4.7 和评注 4.1, 可得如下 Minkowski 不等式的改进形式:

推论 4.3^{[35][13]} 设 $f(x), g(x)$ 是定义在 $[a, b]$ 上的可积函数, 并且 $f(x) > 0$, $g(x) > 0$. 设

$$\begin{aligned} M(t) &= \left[\int_a^t (f(x) + g(x))^{p-1} f(x) dx \right]^2 + \left[\int_a^t (f(x) + g(x))^{p-1} g(x) dx \right]^2 \\ &\quad - \left[\left(\int_a^t f^p(x) dx \right)^{\frac{2}{p}} + \left(\int_a^t g^p(x) dx \right)^{\frac{2}{p}} \right] \left[\int_a^t (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}}. \end{aligned} \quad (4.46)$$

如果 $0 < p < 1$, 则有

$$\begin{aligned} & \left[\int_a^b (f(x) + g(x))^p f(x) dx \right]^2 \\ & \geq \left[\left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} + \left(\int_a^b g^p(x) dx \right)^{\frac{1}{p}} \right]^2 \\ & \quad \cdot \left[\int_a^b (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}} + M\left(\frac{a+b}{2}\right); \end{aligned} \quad (4.47)$$

如果 $p > 1$, 则有

$$\begin{aligned} & \left[\int_a^b (f(x) + g(x))^p f(x) dx \right]^2 \\ & \leq \left[\left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} + \left(\int_a^t g^p(x) dx \right)^{\frac{1}{p}} \right]^2 \\ & \quad \cdot \left[\int_a^b (f(x) + g(x))^p dx \right]^{2-\frac{2}{p}} + M \left(\frac{a+b}{2} \right). \end{aligned} \quad (4.48)$$

第5章

应 用

自从胡克教授给出胡克不等式以及它的复数形式以来,出现了大量的关于这两个不等式的应用.近几年来,随着对这两个不等式研究的深入,一些新的应用又相继出现.本章的主要目的是介绍作者在这些领域的成果,关于其他学者的成果读者可参考文献[13]和[41],或者参考本书附录中的内容.

5.1 Aczél-Popoviciu-Vasić 不等式的推广和改进

在 1956 年, Aczél^[16] 建立了如下重要的不等式:

定理 5.1 如果 a_i, b_i ($i = 1, 2, \dots, n$) 是正数, 并且使得

$$a_1^2 - \sum_{i=2}^n a_i^2 > 0, \quad b_1^2 - \sum_{i=2}^n b_i^2 > 0,$$

则有

$$\left(a_1^2 - \sum_{i=2}^n a_i^2\right) \left(b_1^2 - \sum_{i=2}^n b_i^2\right) \leq \left(a_1 b_1 - \sum_{i=2}^n a_i b_i\right)^2. \quad (5.1)$$

不等式(5.1)就是著名的 Aczél 不等式.

在 1959 年, Popoviciu^[31] 首先给出了上述 Aczél 不等式的如下推广:

定理 5.2 设 $p \geq q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, 并且设 a_i, b_i ($i = 1, 2, \dots, n$) 是正数

且使得 $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$, 则有

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \leq a_1 b_1 - \sum_{i=2}^n a_i b_i. \quad (5.2)$$

随后在 1982 年, Vasić 和 Pečarić^[38] 给出了不等式(5.2)的反向形式:

定理 5.3 设 $q < 0$, $p > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, 并且设 a_i, b_i ($i = 1, 2, \dots, n$) 是正数且使得 $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$, 则有

$$\left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \geq a_1 b_1 - \sum_{i=2}^n a_i b_i. \quad (5.3)$$

这里我们给出不等式(5.2)和(5.3)的推广和改进.

定理 5.4^{[36][41]} 设 $a_i, b_i \geq 0$, $a_1^p - \sum_{i=2}^n a_i^p > 0$, $b_1^q - \sum_{i=2}^n b_i^q > 0$, 并且设

$$1 - e_i + e_j \geq 0 \quad (i, j = 1, 2, \dots, n), \quad \mu = \min \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}, \quad \rho = \max \left\{ \frac{1}{p} + \frac{1}{q}, 1 \right\}.$$

则对于 $p \geq q > 0$, 有

$$\begin{aligned} & \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \\ & \leq n^{1-\mu} \cdot b_1^{1-\frac{q}{p}} \left\{ b_1^{2q} a_1^{2p} - \left[a_1^p \left(b_1^q e_1 + \sum_{i=2}^n b_i^q (e_i - e_1) \right) \right. \right. \\ & \quad \left. \left. - b_1^q \left(a_1^p e_1 + \sum_{i=2}^n a_i^p (e_i - e_1) \right) \right]^2 \right\}^{\frac{1}{2q}} - \sum_{i=2}^n a_i b_i. \end{aligned} \quad (5.4)$$

如果 $q < 0$, $p > 0$, 则有

$$\begin{aligned} & \left(a_1^p - \sum_{i=2}^n a_i^p\right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q\right)^{\frac{1}{q}} \\ & \geq n^{1-\rho} \cdot a_1^{1-\frac{p}{q}} \left\{ a_1^{2p} b_1^{2q} - \left[b_1^q \left(a_1^p e_1 + \sum_{i=2}^n a_i^p (e_i - e_1) \right) \right. \right. \end{aligned}$$

$$-a_1^p \left(b_1^q e_1 + \sum_{i=2}^n b_i^q (e_i - e_1) \right) \Bigg]^2 \Bigg\}^{\frac{1}{2q}} - \sum_{i=2}^n a_i b_i. \quad (5.5)$$

证 在(2.8)和(3.10)中分别令

$$A_1^p = a_1^p - \sum_{i=2}^n a_i^p, \quad B_1^q = b_1^q - \sum_{i=2}^n b_i^q;$$

$$A_i = a_i, \quad B_i = b_i \quad (i = 2, 3, \dots, n),$$

则得到我们要证的结果. \square

特别地, 如果在上述定理中令 $a_1 \neq 0, b_1 \neq 0$ 并且 $\frac{1}{p} + \frac{1}{q} = 1$, 则可以得到不等式(5.2)和(5.3)的如下改进:

推论 5.1 设 $a_1 \neq 0, b_1 \neq 0, \frac{1}{p} + \frac{1}{q} = 1$. 如果 $p \geq q > 0$, 则由定理 5.4 有

$$\begin{aligned} & \left(a_1^p - \sum_{i=2}^n a_i^p \right)^{\frac{1}{p}} \left(b_1^q - \sum_{i=2}^n b_i^q \right)^{\frac{1}{q}} \\ & \leq a_1 b_1 \left\{ 1 - \left(\frac{b_1^q e_1 + \sum_{i=2}^n b_i^q (e_i - e_1)}{b_1^q} - \frac{a_1^p e_1 + \sum_{i=2}^n a_i^p (e_i - e_1)}{a_1^p} \right)^2 \right\}^{\frac{1}{2q}} \\ & \quad - \sum_{i=2}^n a_i b_i. \end{aligned} \quad (5.6)$$

当 $q < 0$ 时, 不等式(5.6)反向.

5.2 Hao Z-C 不等式和 A-G 不等式的改进

经典的 A-G 不等式是指: 如果 $a_j > 0, \lambda_j > 0 (j = 1, 2, \dots, k), p > 0$, 并且 $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 则

$$\prod_{j=1}^k a_j^{\frac{1}{\lambda_j}} \leq \sum_{j=1}^k \frac{a_j}{\lambda_j}. \quad (5.7)$$

而 1990 年数学家 Hao Z-C^[23] 建立的如下不等式是上述算术-几何平均不等式的重要改进:

$$\prod_{j=1}^k a_j^{\frac{1}{\lambda_j}} \leq \left\{ p \int_0^\infty \left[\prod_{j=1}^k (x + a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} dx \right\}^{-\frac{1}{p}} \leq \sum_{j=1}^k \frac{a_j}{\lambda_j}. \quad (5.8)$$

其中 $a_j > 0$, $\lambda_j > 0$ ($j = 1, 2, \dots, k$), $p > 0$ 并且 $\sum_{j=1}^k \frac{1}{\lambda_j} = 1$.

在此, 我们给出上述 Hao Z-C 不等式的一个精美改进.

定理 5.5^[32] 设 $a_j > 0$ ($j = 1, 2, \dots, k$), $p > 0$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$,

$\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且设 $1 - e(x) + e(y) \geq 0$, $\int_0^\infty e(x) dx < \infty$. 则有

$$\begin{aligned} \prod_{j=1}^k a_j^{\frac{1}{\lambda_j}} &\leq \left(\prod_{j=1}^k a_j^{\frac{1}{\lambda_j}} \right) \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_j} R^2(x, e; a_j, p) \right) \right]^{-\frac{1}{p}} \\ &\leq \left\{ p \int_0^\infty \left[\prod_{j=1}^k (x + a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} dx \right\}^{-\frac{1}{p}} \\ &\leq \sum_{j=1}^k \frac{a_j}{\lambda_j}, \end{aligned} \quad (5.9)$$

$$\text{其中 } \rho(k) = \begin{cases} \frac{k}{2}, & \text{若 } k \text{ 是偶数,} \\ \frac{k-1}{2}, & \text{若 } k \text{ 是奇数,} \end{cases}$$

$$R(x, e; a_j, p) = \frac{\int_0^\infty (x + a_{2j-1})^{-p-1} e(x) dx}{\int_0^\infty (x + a_{2j-1})^{-p-1} dx} - \frac{\int_0^\infty (x + a_{2j})^{-p-1} e(x) dx}{\int_0^\infty (x + a_{2j})^{-p-1} dx}.$$

证 对于 $x \geq 0$, 在(5.8)中利用一个代换:

$$a_j \rightarrow x + a_j.$$

则有

$$0 < \prod_{j=1}^k (x + a_j)^{\frac{1}{\lambda_j}} \leq \sum_{j=1}^k \frac{x + a_j}{\lambda_j} = x + \sum_{j=1}^k \frac{a_j}{\lambda_j}. \quad (5.10)$$

在上式两端从 0 到 ∞ 取积分有

$$\begin{aligned} \int_0^\infty \left(\prod_{j=1}^k (x+a_j)^{\frac{1}{\lambda_j}} \right)^{-p-1} dx &\geq \int_0^\infty \left(x + \sum_{j=1}^k \frac{a_j}{\lambda_j} \right)^{-p-1} dx \\ &= \frac{1}{p} \left(\sum_{j=1}^k \frac{a_j}{\lambda_j} \right)^{-p}. \end{aligned} \quad (5.11)$$

另一方面, 应用不等式(2.27)有

$$\begin{aligned} \int_0^\infty \left[\prod_{j=1}^k (x+a_j)^{\frac{1}{\lambda_j}} \right]^{-p-1} dx &= \int_0^\infty \prod_{j=1}^k [(x+a_j)^{-p-1}]^{\frac{1}{\lambda_j}} dx \\ &\leq \left[\prod_{j=1}^k \left(\int_0^\infty (x+a_j)^{-p-1} dx \right)^{\frac{1}{\lambda_j}} \right] \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_{2j}} R^2(x, e; a_j, p) \right) \right] \\ &= \left(\frac{1}{p} \prod_{j=1}^k a_j^{-\frac{p}{\lambda_j}} \right) \left[\prod_{j=1}^{\rho(k)} \left(1 - \frac{1}{2\lambda_{2j}} R^2(x, e; a_j, p) \right) \right]. \end{aligned} \quad (5.12)$$

由不等式(5.12)和(5.11)立刻可得要证的不等式(5.9). \square

由上述定理 5.5 立刻可得如下算术-几何平均不等式的新的改进:

推论 5.2^[32] 设 $a_j > 0$ ($j = 1, 2, \dots, k$), $p > 0$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$,

$\sum_{j=1}^k \frac{1}{\lambda_j} = 1$, 并且设 $1 - e(x) + e(y) \geq 0$, $\int_0^\infty e(x) dx < \infty$. 则有

$$\prod_{j=1}^k a_j^{\frac{1}{\lambda_j}} \leq \left[\prod_{j=2}^{\rho(k)} \left(1 - \frac{1}{2\lambda_j} R^2(x, e; a_j, p) \right) \right]^{\frac{1}{p}} \left(\sum_{j=1}^k \frac{a_j}{\lambda_j} \right), \quad (5.13)$$

$$\text{其中 } \rho(k) = \begin{cases} \frac{k}{2}, & \text{若 } k \text{ 是偶数,} \\ \frac{k-1}{2}, & \text{若 } k \text{ 是奇数,} \end{cases}$$

$$R(x, e; a_j, p) = \frac{\int_0^\infty (x+a_{2j-1})^{-p-1} e(x) dx}{\int_0^\infty (x+a_{2j-1})^{-p-1} dx} - \frac{\int_0^\infty (x+a_{2j})^{-p-1} e(x) dx}{\int_0^\infty (x+a_{2j})^{-p-1} dx}.$$

5.3 Hardy 型不等式的改进^[37]

设 $f(x) \geq 0$, $f(x) \in L^p(0, \infty)$, $p > 1$. 则著名的 Hardy 不等式是指:

$$\int_0^\infty \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy \leq \left(\int_0^\infty K(x, 1) x^{-\frac{1}{p}} dx \right)^p \int_0^\infty f^p(x) dx, \quad (5.14)$$

其中 $K(x, y) \geq 0$ 为齐负一次式. 如果 $0 < p < 1$, 则上述不等式(5.14)反向.

不等式(5.14)的两个重要特例是指:

定理 5.6 设 $p > 1$, $f(x) \geq 0$, 并且对于任意的 $a > 0$, 函数 $f(x)$ 在区间 $[0, a]$ 或 $[a, \infty)$ 上都是 Lebesgue 可积的, 于是

$$\int_0^\infty y^{-r} F^p(y) dy \leq \left(\frac{p}{r-1} \right)^p \int_0^\infty x^{-r} (xf(x))^p dx \quad (r > 1), \quad (5.15)$$

且

$$\int_0^\infty y^{-r} F^p(y) dy \leq \left(\frac{p}{1-r} \right)^p \int_0^\infty x^{-r} (xf(x))^p dx \quad (r < 1), \quad (5.16)$$

其中

$$F(y) = \begin{cases} \int_0^y f(x) dx, & r > 1, \\ \int_y^\infty f(x) dx, & r < 1. \end{cases}$$

如果 $0 < p < 1$, 则上述不等式反向.

1977 年, Imoru 得到了下面一类 Hardy 型不等式:

定理 5.7 设函数 g 在区间 $[0, \infty]$ 上是单调递增且连续的函数, $g(0) = 0$, $g(x) > 0$ ($x > 0$), $g(\infty) = \infty$. 当 $r > 1$ 时, 非负函数 $f(x)$ 在区间 $[0, b]$ 上是 Lebesgue 可积的; 当 $r < 1$ 时, 非负函数 $f(x)$ 在区间 $[a, \infty)$ 上也是 Lebesgue 可积的, 其中 $a, b > 0$. 假设

$$F(x) = \begin{cases} \int_0^x f(t)dg(t), & r > 1, \\ \int_x^\infty f(t)dg(t), & r < 1. \end{cases}$$

如果 $p \geq 1$, 则

$$\begin{aligned} & \int_0^b g^{-r}(x)F^p(x)dg(x) + \frac{p}{r-1}g^{1-r}(b)F^p(b) \\ & \leq \left(\frac{p}{r-1}\right)^p \int_0^b g^{-r}(x)(f(x)g(x))^p dg(x) \quad (r > 1), \end{aligned} \quad (5.17)$$

并且

$$\begin{aligned} & \int_a^\infty g^{-r}(x)F^p(x)dg(x) + \frac{p}{1-r}g^{1-r}(a)F^p(a) \\ & \leq \left(\frac{p}{1-r}\right)^p \int_a^\infty g^{-r}(x)(f(x)g(x))^p dg(x) \quad (r < 1). \end{aligned} \quad (5.18)$$

如果 $0 < p \leq 1$, 则上述所有不等式反向.

引理 5.1 设函数 $g(x)$ 在区间 $[a, b]$ 上是单调递增的连续函数, 并且设函数 $\varphi(x, t), e(x)$ 在区间 $[0, +\infty)$ 上是可积的, 其中 $\varphi(x, t) \geq 0$, $1 - e(x) + e(y) \geq 0$, 且 ϕ 是单调递增函数. 如果 $p \geq 1$, 则

$$\begin{aligned} & \int_0^b g^{-1}(x) \left(\int_0^x \varphi(x, t) d\phi(t) \right) dg(x) \\ & \geq \int_0^b \left\{ g^{-1}(x) \left(\int_0^x \varphi^{\frac{1}{p}}(x, t) d\phi(t) \right)^p \left(\int_0^x d\phi(t) \right)^{1-p} \right. \\ & \quad \cdot \left[1 - \left(\frac{\int_0^x \varphi(x, t) e(t) d\phi(t)}{\int_0^x \varphi(x, t) d\phi(t)} - \frac{\int_0^x e(t) d\phi(t)}{\int_0^x d\phi(t)} \right)^2 \right]^{\frac{p}{2}} \left. \right\} dg(x), \end{aligned} \quad (5.19)$$

并且

$$\begin{aligned} & \int_a^\infty g^{-1}(x) \left(\int_x^\infty \varphi(x, t) d\phi(t) \right) dg(x) \\ & \geq \int_a^\infty \left\{ g^{-1}(x) \left(\int_x^\infty \varphi^{\frac{1}{p}}(x, t) d\phi(t) \right)^p \left(\int_x^\infty d\phi(t) \right)^{1-p} \right. \end{aligned}$$

$$\left\{ \left[1 - \left(\frac{\int_x^\infty \varphi(x, t) e(t) d\phi(t)}{\int_x^\infty \varphi(x, t) d\phi(t)} - \frac{\int_x^\infty e(t) d\phi(t)}{\int_x^\infty d\phi(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x), \quad (5.20)$$

其中 $\beta = \max\{-1, 1-p\}$. 如果 $0 < p < 1$, 则

$$\begin{aligned} & \int_0^b g^{-1}(x) \left(\int_0^x \varphi(x, t) d\phi(t) \right) dg(x) \\ & \leq \int_0^b \left\{ g^{-1}(x) \left(\int_0^x \varphi^{\frac{1}{p}}(x, t) d\phi(t) \right)^p \cdot \left(\int_0^x d\phi(t) \right)^{1-p} \right. \\ & \quad \cdot \left. \left[1 - \left(\frac{\int_0^x \varphi^{\frac{1}{p}}(x, t) e(t) d\phi(t)}{\int_0^x \varphi^{\frac{1}{p}}(x, t) d\phi(t)} - \frac{\int_0^x e(t) d\phi(t)}{\int_0^x d\phi(t)} \right)^2 \right]^{\frac{\gamma}{2}} \right\} dg(x). \end{aligned} \quad (5.21)$$

并且

$$\begin{aligned} & \int_a^\infty g^{-1}(x) \left(\int_x^\infty \varphi(x, t) d\phi(t) \right) dg(x) \\ & \leq \int_a^\infty \left\{ g^{-1}(x) \left(\int_x^\infty \varphi^{\frac{1}{p}}(x, t) d\phi(t) \right)^p \left(\int_x^\infty d\phi(t) \right)^{1-p} \right. \\ & \quad \cdot \left. \left[1 - \left(\frac{\int_x^\infty \varphi^{\frac{1}{p}}(x, t) e(t) d\phi(t)}{\int_x^\infty \varphi^{\frac{1}{p}}(x, t) d\phi(t)} - \frac{\int_x^\infty e(t) d\phi(t)}{\int_x^\infty d\phi(t)} \right)^2 \right]^{\frac{\gamma}{2}} \right\} dg(x). \end{aligned} \quad (5.22)$$

其中 $\gamma = \min\{p, 1-p\}$.

证 由不等式(3.6)和(2.6)易得要证的结论. \square

引理 5.2 设函数 $g(x)$ 在区间 $[0, \infty]$ 上是单调递增且连续的函数,

$g(0) = 0$, $g(x) > 0$ ($x > 0$), $g(\infty) = \infty$. 设 $\delta = \frac{1-r}{p}$, $r \neq 1$, 当 $r > 1$

时, 非负函数 $f(x), e(x)$ 在区间 $[0, b]$ 上关于 $g(x)$ 是 Lebesgue 可积的;

当 $r < 1$ 时, 非负函数 $f(x), e(x)$ 在区间 $[a, \infty)$ 上关于 $g(x)$ 也是 Lebesgue 可积的, 其中 $a, b > 0$, 并且对于所有 $x, y \in [0, +\infty)$ 有 $1 - e(x) + e(y) \geq 0$. 假设

$$\lambda(x) = \begin{cases} \int_0^x (g(t))^{(p-1)(1+\delta)} f^p(t) dg(t), & r > 1, \\ \int_x^\infty (g(t))^{(p-1)(1+\delta)} f^p(t) dg(t), & r < 1. \end{cases}$$

如果 $p \geq 1$, 则

$$\begin{aligned} & \int_0^b g^{\delta-1}(x) \lambda(x) dg(x) \\ & \geq (-\delta^{-1})^{1-p} \int_0^b \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_0^x g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^x g^{-(1+\delta)}(t) dg(t)} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x) \quad (r > 1), \quad (5.23) \end{aligned}$$

并且

$$\begin{aligned} & \int_a^\infty g^{\delta-1}(x) \lambda(x) dg(x) \\ & \geq \delta^{p-1} \int_a^\infty \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_x^\infty g^{-(1+\delta)}(t) e(t) dg(t)}{\int_x^\infty g^{-(1+\delta)}(t) dg(t)} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x) \quad (r < 1), \quad (5.24) \end{aligned}$$

其中

$$F(x) = \begin{cases} \int_0^x f(t) dg(t), & r > 1, \\ \int_x^\infty f(t) dg(t), & r < 1, \end{cases}$$

$\beta = \max\{-1, 1-p\}$. 如果 $0 < p < 1$, 则

$$\begin{aligned} & \int_0^b g^{\delta-1}(x) \lambda(x) dg(x) \\ & \leq (-\delta^{-1})^{1-p} \int_0^b \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_0^x f(t)e(t)dg(t)}{\int_0^x f(t)dg(t)} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)} \right)^2 \right]^{\frac{\gamma}{2}} \right\} dg(x) \quad (r > 1), \end{aligned} \quad (5.25)$$

并且

$$\begin{aligned} & \int_a^\infty g^{\delta-1}(x) \lambda(x) dg(x) \\ & \leq \delta^{p-1} \int_a^\infty \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_x^\infty f(t)e(t)dg(t)}{\int_x^\infty f(t)dg(t)} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\int_x^\infty g^{-(1+\delta)}(t)e(t)dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)dg(t)} \right)^2 \right]^{\frac{\gamma}{2}} \right\} dg(x) \quad (r < 1), \end{aligned} \quad (5.26)$$

其中

$$F(x) = \begin{cases} \int_0^x f(t)dg(t), & r > 1, \\ \int_x^\infty f(t)dg(t), & r < 1, \end{cases}$$

$$\gamma = \min\{p, 1-p\}.$$

证 这里仅仅给出不等式(5.23)的证明, 不等式(5.24), (5.25)和(5.26)的证明类似. 在不等式(5.19)中令

$$\begin{aligned} \varphi(x, t) &= g^\delta(x) (g(t))^{p(1+\delta)} f^p(t), \\ d\phi(t) &= (g(t))^{-(1+\delta)} dg(t). \end{aligned}$$

如果 $r, p > 1$, 则有

$$\begin{aligned}
 & \int_0^b g^{\delta-1}(x) \lambda(x) dg(x) \\
 &= \int_0^b g^{-1}(x) \left(\int_0^x \varphi(x, t) d\phi(t) \right) dg(x) \\
 &\geq \int_0^b \left\{ g^{-1}(x) \left(\int_0^x \varphi^{\frac{1}{p}}(x, t) d\phi(t) \right)^p \left(\int_0^x d\phi(t) \right)^{1-p} \right. \\
 &\quad \cdot \left[1 - \left(\frac{\int_0^x \varphi(x, t) e(t) d\phi(t)}{\int_0^x \varphi(x, t) d\phi(t)} - \frac{\int_0^x e(t) d\phi(t)}{\int_0^x d\phi(t)} \right)^2 \right]^{\frac{\beta}{2}} \Big\} dg(x) \\
 &= \int_0^b \left\{ g^{-1+\delta}(x) \left(\int_0^x f(t) dg(t) \right)^p \left(\int_0^x g^{-(1+\delta)}(t) dg(t) \right)^{1-p} \right. \\
 &\quad \cdot \left[1 - \left(\frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^x g^{-(1+\delta)}(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \Big\} dg(x) \\
 &= (-\delta^{-1})^{1-p} \int_0^b \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{\int_0^x g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^x g^{-(1+\delta)}(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x).
 \end{aligned}$$

这样就得到我们要证的不等式(5.23). □

引理 5.3 设函数 $g(x)$ 在区间 $[0, \infty]$ 上是单调递增且连续的函数,

$g(0) = 0, g(x) > 0 (x > 0), g(\infty) = \infty$. 设 $\delta = \frac{1-r}{p}, r \neq 1$, 当 $r > 1$

时, 非负函数 $f(x), e(x)$ 在区间 $[0, b]$ 上关于 $g(x)$ 是 Lebesgue 可积的; 当 $r < 1$ 时, 非负函数 $f(x), e(x)$ 在区间 $[a, \infty)$ 上关于 $g(x)$ 也是 Lebesgue 可积的, 其中 $a, b > 0$, 并且对于所有 $x, y \in [0, +\infty)$ 有 $1 - e(x) + e(y) \geq 0$. 假设

$$\lambda(x) = \begin{cases} \int_0^x (g(t))^{(p-1)(1+\delta)} f^p(t) dg(t), & r > 1, \\ \int_x^\infty (g(t))^{(p-1)(1+\delta)} f^p(t) dg(t), & r < 1. \end{cases}$$

如果 $p > 1$, 则

$$g^\delta(b)\lambda(b) \geq (-\delta^{-1})^{1-p} g^{\delta p}(b) F^p(b) \left[1 - \left(\frac{\int_0^b g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^b g^{-(1+\delta)}(t) dg(t)} - \frac{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{p}{2}} \quad (r > 1), \quad (5.27)$$

并且

$$g^\delta(a)\lambda(a) \geq \delta^{p-1} g^{\delta p}(a) F^p(a) \left[1 - \left(\frac{\int_a^\infty g^{-(1+\delta)}(t) e(t) dg(t)}{\int_a^\infty g^{-(1+\delta)}(t) dg(t)} - \frac{\int_a^\infty g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_a^\infty g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{p}{2}} \quad (r < 1). \quad (5.28)$$

如果 $0 < p < 1$, 则

$$g^\delta(b)\lambda(b) \leq (-\delta^{-1})^{1-p} g^{\delta p}(b) F^p(b) \left[1 - \left(\frac{\int_0^b f(t) e(t) dg(t)}{\int_0^b f(t) dg(t)} - \frac{\int_0^b g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^b g^{-(1+\delta)}(t) dg(t)} \right)^2 \right]^{\frac{p}{2}} \quad (r > 1), \quad (5.29)$$

并且

$$g^\delta(a)\lambda(a) \leq \delta^{p-1} g^{\delta p}(a) F^p(a)$$

$$\cdot \left[1 - \left(\frac{\int_a^\infty f(t)e(t)dg(t)}{\int_a^\infty f(t)dg(t)} - \frac{\int_a^\infty g^{-(1+\delta)}(t)e(t)dg(t)}{\int_a^\infty g^{-(1+\delta)}(t)dg(t)} \right)^2 \right]^{\frac{\gamma}{2}} \\ (r < 1). \quad (5.30)$$

证 在此仅仅给出不等式(5.27)的证明, 不等式(5.28), (5.29)和(5.30)的证明可以类似地给出. 如果 $r, p > 1$, 则由不等式(5.19)可知

$$\begin{aligned} g^\delta(b)\lambda(b) &= g^\delta(b) \int_0^b g^{(p-1)(1+\delta)}(t) f^p(t) dg(t) \\ &= \int_0^b \varphi(b, t) d\phi(t) \\ &\geq \left(\int_0^b \varphi^{\frac{1}{p}}(b, t) d\phi(t) \right)^p \left(\int_0^b d\phi(t) \right)^{1-p} \\ &\cdot \left[1 - \left(\frac{\int_0^b \varphi(b, t)e(t) d\phi(t)}{\int_0^b \varphi(b, t) d\phi(t)} - \frac{\int_0^b e(t) d\phi(t)}{\int_0^b d\phi(t)} \right)^2 \right]^{\frac{\beta}{2}} \\ &= (-\delta^{-1})^{1-p} g^{\delta p}(b) F^p(b) \left[1 - \left(\frac{\int_0^b g^{-(1+\delta)}(t)e(t) dg(t)}{\int_0^b g^{-(1+\delta)}(t) dg(t)} \right. \right. \\ &\quad \left. \left. - \frac{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^b g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}}, \end{aligned}$$

其中

$$\begin{aligned} \varphi(b, t) &= g^\delta(b) (g(t))^{p(1+\delta)} f^p(t), \\ d\phi(t) &= (g(t))^{-(1+\delta)} dg(t). \end{aligned}$$

这样就得到我们要证的不等式(5.27). □

定理 5.8 设函数 $g(x)$ 在区间 $[0, \infty]$ 上是单调递增且连续的函数,

$g(0) = 0$, $g(x) > 0$ ($x > 0$), $g(\infty) = \infty$. 当 $r > 1$ 时, 非负函数 $f(x)$, $e(x)$ 在区间 $[0, b]$ 上关于 $g(x)$ 是 Lebesgue 可积的; 当 $r < 1$ 时, 非负函数 $f(x), e(x)$ 在区间 $[a, \infty)$ 上关于 $g(x)$ 也是 Lebesgue 可积的, 其中 $a, b > 0$, 并且对于所有 $x, y \in [0, +\infty)$ 有 $1 - e(x) + e(y) \geq 0$. 设

$$F(x) = \begin{cases} \int_0^x f(t)dg(t), & r > 1, \\ \int_x^\infty f(t)dg(t), & r < 1. \end{cases}$$

如果 $p \geq 1$, 则

$$\begin{aligned} & \int_0^b g^{-r}(x) F^p(x) \left(1 - \frac{\beta}{2} \omega^2(f, g, e; x)\right) dg(x) \\ & \quad + \frac{p}{r-1} g^{1-r}(b) F^p(b) \left(1 - \frac{\beta}{2} \omega^2(f, g, e; b)\right) \\ & \leq \left(\frac{p}{r-1}\right)^p \int_0^b g^{-r}(x) (f(x)g(x))^p dg(x) \quad (r > 1), \end{aligned} \quad (5.31)$$

并且

$$\begin{aligned} & \int_a^\infty g^{-r}(x) F^p(x) \left(1 - \frac{\beta}{2} \varpi^2(f, g, e; x)\right) dg(x) \\ & \quad + \frac{p}{1-r} g^{1-r}(b) F^p(b) \left(1 - \frac{\beta}{2} \varpi^2(f, g, e; a)\right) \\ & \leq \left(\frac{p}{1-r}\right)^p \int_0^b g^{-r}(x) (f(x)g(x))^p dg(x) \quad (r < 1), \end{aligned} \quad (5.32)$$

其中 $\beta = \max\{-1, 1-p\}$, $\delta = \frac{1-r}{p}$,

$$\omega(f, g, e; x) = \frac{\int_0^x g^{-(1+\delta)}(t) e(t) dg(t)}{\int_0^x g^{-(1+\delta)}(t) dg(t)} - \frac{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)},$$

$$\varpi(f, g, e; x) = \frac{\int_x^\infty g^{-(1+\delta)}(t) e(t) dg(t)}{\int_x^\infty g^{-(1+\delta)}(t) dg(t)} - \frac{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) e(t) dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)}.$$

如果 $0 < p < 1$, 则

$$\begin{aligned}
 & \int_0^b g^{-r}(x) F^p(x) \left(1 - \frac{\gamma}{2} \mu^2(f, g, e; x)\right) dg(x) \\
 & \quad + \frac{p}{r-1} g^{1-r}(b) F^p(b) \left(1 - \frac{\gamma}{2} \mu^2(f, g, e; b)\right) \\
 & \geq \left(\frac{p}{r-1}\right)^p \int_0^b g^{-r}(x) (f(x)g(x))^p dg(x) \quad (r > 1), \quad (5.33)
 \end{aligned}$$

并且

$$\begin{aligned}
 & \int_a^\infty g^{-r}(x) F^p(x) \left(1 - \frac{\gamma}{2} \nu^2(f, g, e; x)\right) dg(x) \\
 & \quad + \frac{p}{1-r} g^{1-r}(b) F^p(b) \left(1 - \frac{\gamma}{2} \nu^2(f, g, e; a)\right) \\
 & \geq \left(\frac{p}{1-r}\right)^p \int_0^b g^{-r}(x) (f(x)g(x))^p dg(x) \quad (r < 1), \quad (5.34)
 \end{aligned}$$

其中 $\gamma = \min\{p, 1-p\}$, $\delta = \frac{1-r}{p}$,

$$\begin{aligned}
 \mu(f, g, e; x) &= \frac{\int_0^x f(t)e(t)dg(t)}{\int_0^x f(t)dg(t)} - \frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)}, \\
 \nu(f, g, e; x) &= \frac{\int_x^\infty f(t)e(t)dg(t)}{\int_x^\infty f(t)dg(t)} - \frac{\int_x^\infty g^{-(1+\delta)}(t)e(t)dg(t)}{\int_x^\infty g^{-(1+\delta)}(t)dg(t)}.
 \end{aligned}$$

证 这里仅仅证明 $p \geq 1$ 时的情形, $0 < p < 1$ 时的证明类似.

(1) 当 $r > 1$ 时, 由函数 $g(x)$ 单调递增的性质可知

$$\begin{aligned}
 0 < \lambda(x) &= \int_0^x g^{-(1-p)(1+\delta)}(t) f^p(t) dg(t) \\
 &= \int_0^x g^{\frac{r-1}{p}}(t) (g^{p-r}(t) f^p(t)) dg(t) \\
 &\leq g^{\frac{r-1}{p}}(x) \int_0^x g^{p-r}(t) f^p(t) dg(t),
 \end{aligned}$$

进而有

$$\lim_{x \rightarrow 0^+} g^\delta(x) \lambda(x) = 0,$$

利用分部积分公式以及上面这个等式和不等式(5.23)可知

$$\begin{aligned}
 & \int_0^b g^{\delta-1}(x)\lambda(x)dg(x) \\
 &= \delta^{-1}g^{\delta}(b)\lambda(b) - \delta^{-1} \int_0^b g^{\delta p-1}(x)(g(x)f(x))^p dg(x) \\
 &\geq (-\delta^{-1})^{1-p} \int_0^b \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)} \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)e(t)dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x),
 \end{aligned}$$

也就是,

$$\begin{aligned}
 & \int_0^b \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_0^x g^{-(1+\delta)}(t)e(t)dg(t)}{\int_0^x g^{-(1+\delta)}(t)dg(t)} \right. \right. \right. \\
 &\quad \left. \left. \left. - \frac{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)e(t)dg(t)}{\int_0^x g^{(1+\delta)(p-1)}(t)f^p(t)dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x) \\
 &\leq -\left(\frac{p}{r-1}\right)^p g^{\delta}(b)\lambda(b) + \left(\frac{p}{r-1}\right)^p \int_0^b g^{-r}(x)(g(x)f(x))^p dg(x). \quad (5.35)
 \end{aligned}$$

联合不等式(5.27)和(5.35)立刻可得要证的不等式(5.31).

(2) 当 $r < 1$ 时, 采用与(1)的情况类似的方法可得

$$0 < \lambda(x) = \int_x^{\infty} g^{-(1-p)(1+\delta)}(t)f^p(t)dg(t) \leq g^{\frac{r-1}{p}}(x) \int_x^{\infty} g^{p-r}(t)f^p(t)dg(t),$$

进而

$$\lim_{x \rightarrow \infty} g^{\delta}(x)\lambda(x) = 0,$$

利用分部积分公式以及上面这个等式和不等式(5.24)有

$$\begin{aligned}
 & \int_a^{\infty} g^{\delta-1}(x)\lambda(x)dg(x) \\
 &= (-\delta)^{-1}g^{\delta}(a)\lambda(a) + \delta^{-1} \int_a^{\infty} g^{\delta p-1}(x)(g(x)f(x))^p dg(x)
 \end{aligned}$$

$$\geq \delta^{p-1} \int_a^\infty \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_x^\infty g^{-(1+\delta)}(t) \epsilon(t) dg(t)}{\int_x^\infty g^{-(1+\delta)}(t) dg(t)} - \frac{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) \epsilon(t) dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x),$$

也就是,

$$\begin{aligned} & \int_a^\infty \left\{ (g(x))^{\delta p-1} F^p(x) \left[1 - \left(\frac{\int_x^\infty g^{-(1+\delta)}(t) \epsilon(t) dg(t)}{\int_x^\infty g^{-(1+\delta)}(t) dg(t)} - \frac{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) \epsilon(t) dg(t)}{\int_x^\infty g^{(1+\delta)(p-1)}(t) f^p(t) dg(t)} \right)^2 \right]^{\frac{\beta}{2}} \right\} dg(x) \\ & \leq -\left(\frac{p}{1-r}\right)^p g^\delta(b) \lambda(b) + \left(\frac{p}{1-r}\right)^p \int_0^b g^{-r}(x) (g(x) f(x))^p dg(x). \end{aligned} \quad (5.36)$$

联合不等式(5.28)和(5.36)立刻可得要证的不等式(5.32).

5.4 Minkowski 不等式的改进^[33]

定理 5.9 设 $a_k > 0$, $b_k > 0$ ($k = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1,$

$2, \dots, n$). 如果 $0 < p < 1$, $\lambda = \max \left\{ -1, 1 - \frac{1}{p} \right\}$, 则有

$$\begin{aligned} \left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{1}{p}} & \geq \left[\left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \right] \left(1 - \frac{\lambda}{2} \varpi^2(a, b, e) \right) \\ & \geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}}, \end{aligned} \quad (5.37)$$

其中

$$\varpi(a, b, e) = \frac{1}{\left[\sum_{k=1}^n (a_k + b_k)^p \right]^2} \left[\sum_{k=1}^n (a_k + b_k)^p e_k \sum_{k=1}^n a_k (a_k + b_k)^{p-1} - \sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k \right].$$

积分形式如下:

定理 5.10 设 $f(x), g(x), e(x)$ 是定义在区间 $[a, b]$ 上的可积函数, 并且 $f(x) > 0, g(x) > 0$. 对于任意的 $x, y \in [a, b]$ 有 $1 - e(x) + e(y) \geq 0$. 如果 $0 < p < 1, \lambda = \max \left\{ -1, 1 - \frac{1}{p} \right\}$, 则有

$$\begin{aligned} & \left[\int_a^b (f(x) + g(x))^p dx \right]^{\frac{1}{p}} \\ & \geq \left[\left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} + \left(\int_a^b g^p(x) dx \right)^{\frac{1}{p}} \right] \left(1 - \frac{\lambda}{2} \omega^2(a, b, e) \right) \\ & \geq \left(\int_a^b f^p(x) dx \right)^{\frac{1}{p}} + \left(\int_a^b g^p(x) dx \right)^{\frac{1}{p}}, \end{aligned} \quad (5.38)$$

其中

$$\begin{aligned} \omega(a, b, e) = & \frac{1}{\left[\int_a^b (f(x) + g(x))^p dx \right]^2} \\ & \cdot \left[\int_a^b (f(x) + g(x))^p e(x) dx \int_a^b f(x) (f(x) + g(x))^{p-1} dx \right. \\ & \left. - \int_a^b (f(x) + g(x))^p dx \int_a^b f(x) (f(x) + g(x))^{p-1} e(x) dx \right]. \end{aligned} \quad (5.39)$$

证 这里仅仅给出定理 5.9 的证明, 定理 5.10 的证明类似. 记

$$\sum_{k=1}^n (a_k + b_k)^p = \sum_{k=1}^n a_k (a_k + b_k)^{p-1} + \sum_{k=1}^n b_k (a_k + b_k)^{p-1}.$$

由推论 3.8 可得

$$\begin{aligned}
 \sum_{k=1}^n (a_k + b_k)^p &\geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{p-1}{p}} \\
 &\quad \cdot \left[1 - \frac{\lambda}{2} \left(\frac{\sum_{k=1}^n (a_k + b_k)^p e_k}{\sum_{k=1}^n (a_k + b_k)^p} - \frac{\sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k}{\sum_{k=1}^n a_k (a_k + b_k)^{p-1}} \right)^2 \right] \\
 &\quad + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{p-1}{p}} \\
 &\quad \cdot \left[1 - \frac{\lambda}{2} \left(\frac{\sum_{k=1}^n (a_k + b_k)^p e_k}{\sum_{k=1}^n (a_k + b_k)^p} - \frac{\sum_{k=1}^n b_k (a_k + b_k)^{p-1} e_k}{\sum_{k=1}^n b_k (a_k + b_k)^{p-1}} \right)^2 \right]. \quad (5.40)
 \end{aligned}$$

上式两端同除以 $\left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{p-1}{p}}$, 有

$$\begin{aligned}
 \left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{1}{p}} &\geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \\
 &\quad - \frac{\lambda}{2} \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left(\frac{\sum_{k=1}^n (a_k + b_k)^p e_k}{\sum_{k=1}^n (a_k + b_k)^p} - \frac{\sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k}{\sum_{k=1}^n a_k (a_k + b_k)^{p-1}} \right)^2 \\
 &\quad - \frac{\lambda}{2} \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \left(\frac{\sum_{k=1}^n (a_k + b_k)^p e_k}{\sum_{k=1}^n (a_k + b_k)^p} - \frac{\sum_{k=1}^n b_k (a_k + b_k)^{p-1} e_k}{\sum_{k=1}^n b_k (a_k + b_k)^{p-1}} \right)^2. \quad (5.41)
 \end{aligned}$$

另一方面,

$$\sum_{k=1}^n (a_k + b_k)^p e_k = \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k + \sum_{k=1}^n b_k (a_k + b_k)^{p-1} e_k. \quad (5.42)$$

进而, 由不等式(5.41)和(5.42)有

$$\begin{aligned}
 & \left[\sum_{k=1}^n (a_k + b_k)^p \right]^{\frac{1}{p}} \\
 & \geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} + \frac{(-\lambda) \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}}}{2 \left[\sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} \right]^2} \\
 & \quad \cdot \left[\sum_{k=1}^n (a_k + b_k)^p e_k \sum_{k=1}^n a_k (a_k + b_k)^{p-1} \right. \\
 & \quad \left. - \sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k \right]^2 \\
 & \quad + \frac{(-\lambda) \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}}}{2 \left[\sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n b_k (a_k + b_k)^{p-1} \right]^2} \\
 & \quad \cdot \left[\sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k \right. \\
 & \quad \left. - \sum_{k=1}^n a_k (a_k + b_k)^{p-1} \sum_{k=1}^n (a_k + b_k)^p e_k \right]^2 \\
 & \geq \left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} + \frac{(-\lambda) \left[\left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \right]}{2 \left[\sum_{k=1}^n (a_k + b_k)^p \right]^4} \\
 & \quad \cdot \left[\sum_{k=1}^n (a_k + b_k)^p e_k \sum_{k=1}^n a_k (a_k + b_k)^{p-1} \right. \\
 & \quad \left. - \sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k \right]^2 \\
 & = \left[\left(\sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n b_k^p \right)^{\frac{1}{p}} \right] \left\{ 1 - \frac{\lambda}{2 \left[\sum_{k=1}^n (a_k + b_k)^p \right]^4} \right. \\
 & \quad \cdot \left[\sum_{k=1}^n (a_k + b_k)^p e_k \sum_{k=1}^n a_k (a_k + b_k)^{p-1} \right.
 \end{aligned}$$

$$- \sum_{k=1}^n (a_k + b_k)^p \sum_{k=1}^n a_k (a_k + b_k)^{p-1} e_k \Big]^2 \Big\}. \quad (5.43)$$

从而得到不等式(5.37)的证明. \square

5.5 Wang C-L 不等式的改进

1983 年, Wang, C-L 在[39]中建立了如下重要的不等式:

定理 5.11 设函数 $f(x), g(x)$ 是定义在区间 $[0, T]$ 上的正的可积函数, 并

且设 $\frac{1}{p} + \frac{1}{q} = 1$. 如果 $0 < p < 1$, 则对于任意正数 a, b, c 有

$$\frac{\left(a + c \int_0^T h^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T h(x)g(x) dx} \geq \frac{\left(a + c \int_0^T f^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T f(x)g(x) dx} \quad (5.44)$$

成立, 其中 $h(x) = \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$. 如果 $p > 1$, 则上述不等式 (5.44) 反向.

定理 5.12^[32] 设函数 $f(x), g(x), e(x)$ 是定义在区间 $[0, T]$ 上的可积函数,

并且 $f(x), g(x) > 0$, $1 - e(x) + e(y) \geq 0$. 如果 $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$,

则对于任意正数 a, b, c 有不等式

$$\frac{\left(a + c \int_0^T h^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T h(x)g(x) dx} \geq \frac{\left(a + c \int_0^T f^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T f(x)g(x) dx} \cdot \left[1 - \frac{1}{2q} \left(\frac{c \int_0^T f^p(x) dx}{a + c \int_0^T f^p(x) dx} - \frac{c \int_0^T g^q(x) dx}{a^{-\frac{q}{p}} b^q + c \int_0^T g^q(x) dx} \right)^2 \right] \quad (5.45)$$

成立, 其中 $h(x) = \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$.

证 一方面, 经过一些简单的运算有

$$\frac{\left(a + c \int_0^T h^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T h(x)g(x) dx} = \left(a^{-\frac{q}{p}}b^q + c \int_0^T g^q(x) dx\right)^{-\frac{1}{q}}. \quad (5.46)$$

另一方面, 在不等式 (5.35) 中令 $e_1 = 0$, $e_2 = 1$, $m = 2$, 由推论 3.10 可得

$$\begin{aligned} & b + c \int_0^T f(x)g(x) dx \\ & \geq b + c \left(\int_0^T f^p(x) dx\right)^{\frac{1}{p}} \left(\int_0^T g^q(x) dx\right)^{\frac{1}{q}} \\ & = a^{\frac{1}{p}}(ba^{-\frac{1}{p}}) + \left(c \int_0^T f^p(x) dx\right)^{\frac{1}{p}} \left(c \int_0^T g^q(x) dx\right)^{\frac{1}{q}} \\ & \geq \left(a + c \int_0^T f^p(x) dx\right)^{\frac{1}{p}} \left(a^{-\frac{q}{p}}b^q + c \int_0^T g^q(x) dx\right)^{\frac{1}{q}} \\ & \quad \cdot \left[1 - \frac{1}{2q} \left(\frac{c \int_0^T f^p(x) dx}{a + c \int_0^T f^p(x) dx} - \frac{c \int_0^T g^q(x) dx}{a^{-\frac{q}{p}}b^q + c \int_0^T g^q(x) dx}\right)^2\right], \end{aligned} \quad (5.47)$$

也就是,

$$\begin{aligned} & \left(a^{-\frac{q}{p}}b^q + c \int_0^T g^q(x) dx\right)^{-\frac{1}{q}} \geq \frac{\left(a + c \int_0^T f^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_0^T f(x)g(x) dx} \\ & \quad \cdot \left[1 - \frac{1}{2q} \left(\frac{c \int_0^T f^p(x) dx}{a + c \int_0^T f^p(x) dx} - \frac{c \int_0^T g^q(x) dx}{a^{-\frac{q}{p}}b^q + c \int_0^T g^q(x) dx}\right)^2\right]. \end{aligned} \quad (5.48)$$

联合上述不等式(5.46)和(5.48), 立刻可得要证的不等式(5.45). \square

5.6 Wang C-L 不等式和 Beckenbach 型不等式的时间标度形式

在这一节, 我们首先给出 Wang C-L 不等式的时间标度版本及其改进形式, 进而得到 Beckenbach 型不等式的时间标度版本及其改进形式.

经典的 Beckenbach 不等式是指:

定理 5.13^[19] 设 $0 \leq p \leq 1$, 并且 $a_i, b_i > 0$ ($i = 1, 2, \dots, n$). 则有

$$\frac{\sum_{i=1}^n (a_i + b_i)^p}{\sum_{i=1}^n (a_i + b_i)^{p-1}} \geq \frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}}.$$

Wang C-L 不等式是指:

定理 5.14^[39] 设函数 $f(x), g(x)$ 是定义在区间 $[s, t]$ 上的正的可积函数,

并且设 $\frac{1}{p} + \frac{1}{q} = 1$. 如果 $0 < p < 1$, 则对于任意正数 a, b, c 有

$$\frac{\left(a + c \int_s^t k^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_s^t k(x)g(x)dx} \geq \frac{\left(a + c \int_s^t f^p(x) dx\right)^{\frac{1}{p}}}{b + c \int_s^t f(x)g(x)dx} \quad (5.49)$$

成立, 其中 $k(x) = \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$. 如果 $p > 1$, 则不等式(5.49)反向.

定理 5.15^[36] 设 $f(x), g(x), h(x) \in C_{rd}([s, t], [0, +\infty))$, 并且设 $\frac{1}{p} + \frac{1}{q} = 1$.

如果 $p > 1$, 则对于任意正数 a, b, c 有不等式

$$\frac{\left(a + c \int_s^t h(x) k^p(x) \Delta x\right)^{\frac{1}{p}}}{b + c \int_s^t h(x) k(x) g(x) \Delta x} \leq \frac{\left(a + c \int_s^t h(x) f^p(x) \Delta x\right)^{\frac{1}{p}}}{b + c \int_s^t h(x) f(x) g(x) \Delta x} \cdot \left[1 - \frac{1}{2q} \left(\frac{c \int_s^t h(x) f^p(x) \Delta x}{a + c \int_s^t h(x) f^p(x) \Delta x} - \frac{c \int_s^t h(x) g^q(x) \Delta x}{a^{-\frac{q}{p}} b^q + c \int_s^t h(x) g^q(x) \Delta x} \right)^2 \right] \quad (5.50)$$

成立, 其中 $k(x) = \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}}$. 如果 $0 < p < 1$, 则不等式(5.50)反向.

证 这里仅仅考虑 $0 < p < 1$ 时的情形, $p > 1$ 时的情形类似. 注意到 $1 + \frac{q}{p} = q$, 不等式(5.50)的左边变为

$$\begin{aligned} & \frac{\left[a + c \int_s^t h(x) \left(\frac{ag(x)}{b}\right)^q \Delta x\right]^{\frac{1}{p}}}{b + c \int_s^t h(x) \left(\frac{ag(x)}{b}\right)^{\frac{q}{p}} g(x) \Delta x} \\ &= \frac{\left(\frac{a}{b}\right)^{\frac{q}{p}} \left[a \left(\frac{b}{a}\right)^q + c \int_s^t h(x) g^q(x) \Delta x\right]^{\frac{1}{p}}}{\left(\frac{a}{b}\right)^{\frac{q}{p}} \left[b \left(\frac{b}{a}\right)^{\frac{q}{p}} + c \int_s^t h(x) g^q(x) \Delta x\right]} \\ &= \left(a^{-\frac{q}{p}} b^q + c \int_s^t h(x) g^q(x) \Delta x\right)^{-\frac{1}{q}}. \end{aligned} \quad (5.51)$$

另一方面, 利用 Hölder 不等式及(5.45), 其中取 $e_1 = 0$, $e_2 = 1$, 有

$$\begin{aligned} & b + c \int_s^t h(x) f(x) g(x) \Delta x \\ & \geq b + c \left(\int_s^t h(x) f^p(x) \Delta x \right)^{\frac{1}{p}} \left(\int_s^t h(x) g^q(x) \Delta x \right)^{\frac{1}{q}} \\ & = a^{\frac{1}{p}} \left(b a^{-\frac{1}{p}} \right) + \left(c \int_s^t h(x) f^p(x) \Delta x \right)^{\frac{1}{p}} \left(c \int_s^t h(x) g^q(x) \Delta x \right)^{\frac{1}{q}} \end{aligned}$$

$$\geq \left(a + c \int_s^t h(x) f^p(x) \Delta x \right)^{\frac{1}{p}} \left(a^{-\frac{q}{p}} b^q + c \int_s^t h(x) g^q(x) \Delta x \right)^{\frac{1}{q}} \cdot \left[1 - \frac{1}{2q} \left(\frac{c \int_s^t h(x) f^p(x) \Delta x}{a + c \int_s^t h(x) f^p(x) \Delta x} - \frac{c \int_s^t h(x) g^q(x) \Delta x}{a^{-\frac{q}{p}} b^q + c \int_s^t h(x) g^q(x) \Delta x} \right)^2 \right]. \quad (5.52)$$

联合不等式(5.51)和(5.52)立刻可得要证的不等式(5.50).

在不等式(5.50)中取

$$\frac{c \int_s^t h(x) f^p(x) \Delta x}{a + c \int_s^t h(x) f^p(x) \Delta x} = \frac{c \int_s^t h(x) g^q(x) \Delta x}{a^{-\frac{q}{p}} b^q + c \int_s^t h(x) g^q(x) \Delta x},$$

则由定理 5.15 立刻可得 Beckenbach 型不等式的时间标度版本:

推论 5.3^[36] 设 $f(x), g(x), h(x) \in C_{rd}([s, t], [0, +\infty))$, 并且设 $\frac{1}{p} + \frac{1}{q} = 1$. 如

果 $p > 1$, 则对于任意的正数 a, b, c 不等式

$$\frac{\left(a + c \int_s^t h(x) k^p(x) \Delta x \right)^{\frac{1}{p}}}{b + c \int_s^t h(x) k(x) g(x) \Delta x} \leq \frac{\left(a + c \int_s^t h(x) f^p(x) \Delta x \right)^{\frac{1}{p}}}{b + c \int_s^t h(x) f(x) g(x) \Delta x} \quad (5.53)$$

成立, 其中 $k(x) = \left(\frac{ag(x)}{b} \right)^{\frac{q}{p}}$. 如果 $0 < p < 1$, 则不等式(5.53)反向.

5.7 离散型 Beckenbach 不等式的改进

在 1950 年, Beckenbach^[17] 建立了如下重要的不等式:

定理 5.16 设 $0 \leq p \leq 1$, 并且 $a_i, b_i > 0$ ($i = 1, 2, \dots, n$). 则有

$$\frac{\sum_{i=1}^n (a_i + b_i)^p}{\sum_{i=1}^n (a_i + b_i)^{p-1}} \geq \frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}}.$$

利用前面给出的反向胡克不等式可以得到上述离散型 Beckenbach 不等式的如下改进:

定理 5.17^[33] 设 $a_i, b_i > 0$ ($i = 1, 2, \dots, n$), $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$),

并且 $0 < p < 1$, $\lambda = \max \left\{ -1, 1 - \frac{1}{p} \right\}$. 则有

$$\begin{aligned} \frac{\sum_{i=1}^n (a_i + b_i)^p}{\sum_{i=1}^n (a_i + b_i)^{p-1}} &\geq \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}} \right) \\ &\quad \cdot \left\{ 1 - \frac{\lambda}{2} \left[\frac{e_1 \left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + e_2 \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}}{\left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}} \right. \right. \\ &\quad \left. \left. - \frac{e_1 \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + e_2 \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}} \right]^2 \right\} \\ &\geq \frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}}. \end{aligned} \tag{5.54}$$

证 利用 Minkowski 不等式和推论 3.8, 有

$$\sum_{i=1}^n (a_i + b_i)^p \geq \left[\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}} \right]^p$$

$$\begin{aligned}
 &= \left[\left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p}} + \left(\frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p}} \right]^p \\
 &\geq \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}} \right) \left[\left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}} \right]^{p-1} \\
 &\quad \cdot \left\{ 1 - \frac{\lambda}{2} \left[\frac{e_1 \left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + e_2 \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}}{\left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}} \right. \right. \\
 &\quad \left. \left. - \frac{e_1 \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + e_2 \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}} \right]^2 \right\} \\
 &\geq \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}} \right) \left[\sum_{i=1}^n (a_i + b_i)^{p-1} \right] \\
 &\quad \cdot \left\{ 1 - \frac{\lambda}{2} \left[\frac{e_1 \left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + e_2 \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}}{\left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}} \right. \right. \\
 &\quad \left. \left. - \frac{e_1 \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + e_2 \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}} \right]^2 \right\},
 \end{aligned}$$

也就是,

$$\begin{aligned}
 \frac{\sum_{i=1}^n (a_i + b_i)^p}{\sum_{i=1}^n (a_i + b_i)^{p-1}} &\geq \left(\frac{\sum_{i=1}^n a_i^p}{\sum_{i=1}^n a_i^{p-1}} + \frac{\sum_{i=1}^n b_i^p}{\sum_{i=1}^n b_i^{p-1}} \right) \\
 &\cdot \left\{ 1 - \frac{\lambda}{2} \left[\frac{e_1 \left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + e_2 \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}}{\left(\sum_{i=1}^n a_i^{p-1} \right)^{\frac{1}{p-1}} + \left(\sum_{i=1}^n b_i^{p-1} \right)^{\frac{1}{p-1}}} \right. \right. \\
 &\quad \left. \left. - \frac{e_1 \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + e_2 \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}}{\left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n b_i^p \right)^{\frac{1}{p}}} \right]^2 \right\}. \quad (5.55)
 \end{aligned}$$

这样就得到我们要证的不等式.

□

附 录

为了体系的完整性,在本附录中给出胡克教授利用胡克不等式得到的一些经典不等式的改进结果,有关细节的证明读者可以参考文献[13].

Dresher 不等式 设 $A_i, B_i \geq 0$, $1 - e_i + e_j > 0$ ($i, j = 1, 2, \dots, n$), $0 < r < 1 < p$, 则有

$$\left(\frac{\sum_{i=1}^n (A_i + B_i)^p}{\sum_{i=1}^n (A_i + B_i)^r} \right)^{\frac{1}{p-r}} \leq \left[\left(\frac{\sum_{i=1}^n A_i^p}{\sum_{i=1}^n A_i^r} \right)^{\frac{1}{p-r}} + \left(\frac{\sum_{i=1}^n B_i^p}{\sum_{i=1}^n B_i^r} \right)^{\frac{1}{p-r}} \right].$$

相应的 Dresher 不等式的改进如下:

定理 1 设 $A_i, B_i \geq 0$, $1 - e_i + e_j \geq 0$ ($i, j = 1, 2, \dots, n$), $0 < r < 1 < p$, 则有

$$\left(\frac{\sum_{i=1}^n (A_i + B_i)^p}{\sum_{i=1}^n (A_i + B_i)^r} \right)^{\frac{1}{p-r}} \leq \left[\left(\frac{\sum_{i=1}^n A_i^p}{\sum_{i=1}^n A_i^r} \right)^{\frac{1}{p-r}} + \left(\frac{\sum_{i=1}^n B_i^p}{\sum_{i=1}^n B_i^r} \right)^{\frac{1}{p-r}} \right].$$

$$- \left\{ \frac{\left[\left(\sum_{i=1}^n A_i^p \right)^{\frac{1}{p}} + \left(\sum_{i=1}^n B_i^p \right)^{\frac{1}{p}} \right]^r}{\sum_{i=1}^n (A_i + B_i)^r} \right\}^{\frac{1}{p-r}} \cdot \frac{\mu(p)}{2} S^2(A, B, e),$$

其中

$$S(A, B, e) = \left[\frac{\sum_{i=1}^n (A_i^p + B_i^p) e_i}{\sum_{i=1}^n (A_i^p + B_i^p)} - \frac{\sum_{i=1}^n (A_i + B_i)^p e_i}{\sum_{i=1}^n (A_i + B_i)^p} \right] \left[\sum_{i=1}^n (A_i^p + B_i^p) \right]^{\frac{1}{p}},$$

$$\mu(p) = \min \left\{ \frac{1}{2p}, \frac{1}{2} - \frac{1}{2p} \right\}.$$

Nagy 不等式 设 $b > a > 0$, $p > 1$, $q = 1 + \frac{p-1}{p}\alpha$, $f' \in L^p(a, b)$, 并且

$f(x)$ 在区间 $[a, b]$ 内有零点, 则有

$$|f(a)|^q + |f(b)|^q \leq q \left[\left(\int_a^b |f'(x)|^p dx \right)^{\frac{1}{p}} \left(\int_a^b |f(x)|^\alpha dx \right)^{1-\frac{1}{p}} \right].$$

相应的 Nagy 不等式的改进如下:

定理 2 设 $b > a > 0$, $p > 1$, $q = 1 + \frac{p-1}{p}\alpha$, $f' \in L^p(a, b)$, 并且 $f(x)$ 在区

间 $[a, b]$ 内有零点, 此外对于任意 $x, y \in [a, b]$, 函数 $1 - e(x) + e(y) \geq 0$. 则

(1) 当 $1 < p \leq 2$ 时, 有

$$|f(a)|^q + |f(b)|^q \leq q \left(\int_a^b |f'(x)|^p dx \right)^{\frac{2}{p}-1} \cdot \left[\left(\int_a^b |f(x)|^\alpha dx \int_a^b |f'(x)|^p dx \right)^2 - \beta^2 \right]^{\frac{p-1}{2p}};$$

(2) 当 $p > 2$ 时有

$$|f(a)|^q + |f(b)|^q \leq q \left(\int_a^b |f(x)|^\alpha dx \right)^{1-\frac{2}{p}} \cdot \left[\left(\int_a^b |f(x)|^\alpha dx \int_a^b |f'(x)|^p dx \right)^2 - \beta^2 \right]^{\frac{1}{2p}},$$

其中

$$\begin{aligned} \beta &= \int_a^b |f'(x)|^p dx \int_a^b |f(x)|^\alpha e(x) dx \\ &\quad - \int_a^b |f(x)|^\alpha dx \int_a^b |f'(x)|^p e(x) dx. \end{aligned}$$

Opial-beesack 不等式 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数,

$f(0) = 0$, 设 $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, 并且设 $g(x) > 0$, $\int_0^h g^p(x) |f'(x)|^p dx$

和 $\int_0^h g^{-q}(x) dx$ 都存在, 则有

$$\int_0^h |f(x)f'(x)| dx \leq \frac{1}{2} \left(\int_0^h g^p(x) |f'(x)|^p dx \right)^{\frac{2}{p}} \left(\int_0^h g^{-q}(x) dx \right)^{\frac{2}{q}}.$$

相应的 Opial-beesack 不等式的改进如下:

定理 3 (改进 1) 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数, $f(0) = 0$, 设

$p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 - e(x) + e(y) \geq 0$, $g(x) > 0$, $\int_0^h g^p(x) |f'(x)|^p dx$

和 $\int_0^h g^{-q}(x) dx$ 都存在, 则

(1) 当 $p \geq 2$ 时, 有

$$\begin{aligned} \int_0^h |f(x)f'(x)| dx &\leq \frac{1}{2} \left(\int_0^h g^{-q}(x) dx \right)^{\frac{2}{q}-\frac{2}{p}} \\ &\quad \cdot \left[\left(\int_0^h g^p(x) |f'(x)|^p dx \int_0^h g^{-q}(x) dx \right)^2 - \gamma^2(0, h) \right]; \end{aligned}$$

(2) 当 $1 < p < 2$ 时, 有

$$\int_0^h |f(x)f'(x)|dx \leq \frac{1}{2} \left(\int_0^h g^p(x) |f'(x)|^p dx \right)^{\frac{2}{p}-\frac{2}{q}} \cdot \left[\left(\int_0^h g^p(x) |f'(x)|^p dx \int_0^h g^{-q}(x) dx \right)^2 - \gamma^2(0, h) \right]^{\frac{1}{q}},$$

其中

$$\begin{aligned} \gamma(0, h) &= \int_0^h g^{-q}(x) dx \int_0^h e(x)g(x) |f'(x)|^p dx \\ &\quad - \int_0^h g(x) |f'(x)|^p dx \int_0^h e(x)g^{-q}(x) dx. \end{aligned}$$

特别地, 若取 $p=2$, $g(x) \equiv 1$, $e(x) = \frac{1}{2} \cos \frac{\pi x}{h}$, 则有

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{2} \left[\left(\int_0^h |f'(x)|^2 dx \right)^2 - \left(\frac{1}{2} \int_0^h |f'(x)|^2 \cos \frac{\pi x}{h} dx \right)^2 \right]^{\frac{1}{2}}.$$

定理4 (改进2) 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数, $f(0) = f(h) = 0$,

设 $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 - e(x) + e(y) \geq 0$, 并且

$$\int_0^{\frac{h}{2}} e(x)g^{-q}(x)dx = \int_{\frac{h}{2}}^h e(x)g^{-q}(x)dx,$$

设 $g(x) > 0$, 并且 $\int_0^{\frac{h}{2}} g^{-q}(x)dx = \int_{\frac{h}{2}}^h g^{-q}(x)dx$, 则当 $1 < p < 2$ 时有

$$\int_0^h |f(x)f'(x)|dx \leq \frac{1}{2} \left(\int_0^h g^p(x) |f'(x)|^p dx \right)^{\frac{4}{p}-2} \cdot \left[\left(\int_0^h g^p(x) |f'(x)|^p dx \int_0^{\frac{h}{2}} g^{-q}(x) dx \right)^2 - \gamma^2(0, h) \right]^{\frac{1}{q}},$$

其中

$$\begin{aligned} \gamma(0, h) &= \int_0^{\frac{h}{2}} g^{-q}(x) dx \int_0^h e(x)g^p(x) |f'(x)|^p dx \\ &\quad - \int_0^h g(x) |f'(x)|^p dx \int_0^{\frac{h}{2}} g^{-q}(x) dx. \end{aligned}$$

特别地, 若取 $p=2$, $g(x) \equiv 1$, $e(x) = \frac{1}{2} \cos \frac{2\pi x}{h}$, 则有

$$\int_0^h |f(x)f'(x)|dx \leq \frac{h}{4} \left[\left(\int_0^h |f'(x)|^2 dx \right)^2 - \left(\frac{1}{2} \int_0^h |f'(x)|^2 \cos \frac{2\pi x}{h} dx \right)^2 \right]^{\frac{1}{2}}.$$

定理 5 (改进 3) 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数, $f(0) = 0$, 设

$$p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, 1 - e(x) + e(y) \geq 0, g(x) > 0, \int_0^h g^p(x) |f'(x)|^p dx$$

和 $\int_0^h g^{-q}(x) dx$ 都存在, 则当 $0 \leq h_1 < h$ 时有

$$\begin{aligned} & \int_{h_1}^h |f(x)f'(x)|dx \\ & \leq \frac{1}{2} \left(\int_0^h g^{-q}(x) dx \right)^{\frac{2}{q}} \left(\int_0^h g^p(x) |f'(x)|^p dx \right)^{\frac{2}{p}} (1 - \lambda^2(e, f'(x), 0, h))^{\mu(p)} \\ & \quad - \frac{1}{2} \left(\int_0^{h_1} g^{-q}(x) dx \right)^{\frac{2}{q}} \left(\int_0^{h_1} g^p(x) |f'(x)|^p dx \right)^{\frac{2}{p}} \\ & \quad \cdot (1 - \lambda^2(e, f'(x), 0, h_1))^{\mu(p)}, \end{aligned}$$

其中

$$\begin{aligned} \lambda(e, f'(x), 0, t) &= t g^{-q}(x) \int_0^t e(x) g^p(x) |f(x)|^p dx \\ &\quad - \frac{t g^p(x) |f'(x)|^p \int_0^t g^{-q}(x) e(x) dx}{\int_0^t g^{-q}(x) dx \int_0^t g^p(x) |f'(x)|^p dx}, \\ \mu(p) &= \begin{cases} \frac{1}{p}, & p \geq 2, \\ 1 - \frac{1}{p}, & 1 < p < 2. \end{cases} \end{aligned}$$

钟开莱不等式 设 $B_i (i = 1, 2, \dots, n)$ 为任意实数, $A_1 \geq A_2 \geq \dots \geq A_n \geq 0$,

$$\text{且有 } \sum_{i=1}^k A_i \leq \sum_{i=1}^k B_i, \quad k = 1, 2, \dots, n, \text{ 则 } \sum_{i=1}^n A_i^2 \leq \sum_{i=1}^n B_i^2.$$

相应的钟开莱不等式的改进如下:

定理 6 设 $B_i (i = 1, 2, \dots, n)$ 为任意实数, $A_1 \geq A_2 \geq \dots \geq A_n \geq 0$, 且有

$$\sum_{i=1}^k A_i \leq \sum_{i=1}^k B_i, \quad k = 1, 2, \dots, n, \quad \text{设 } 1 - e_i + e_j > 0 \quad (i, j = 1, 2, \dots, n),$$

则

$$\sum_{i=1}^n A_i^p \leq \sum_{i=1}^n |B_i|^p \left[1 - \left(\frac{\sum_{i=1}^n A_i^p e_i \sum_{j=1}^n |B_j|^p - \sum_{i=1}^n A_i^p \sum_{j=1}^n |B_j|^p e_j}{\sum_{i=1}^n A_i^p \sum_{j=1}^n |B_j|^p} \right)^2 \right]^{\frac{\eta(p)}{2}},$$

$$\text{其中 } \eta(p) = \begin{cases} p-1, & p \geq 2, \\ 1, & p < 2. \end{cases}$$

Ky Fan 不等式 设 A, B 和 C 是三个实的 n 阶正定矩阵, $0 \leq \lambda \leq 1$, 则有

$$\frac{1}{|\lambda A + (1-\lambda)B|} \leq \frac{1}{|A|^\lambda |B|^{1-\lambda}}.$$

相应的 Ky Fan 不等式的改进如下:

定理 7(改进 1) 设 A, B 和 C 是三个实的 n 阶正定矩阵, $0 \leq \lambda \leq 1$, 则有

$$\frac{1}{|\lambda A + (1-\lambda)B|} \leq \frac{1}{|A|^\lambda |B|^{1-\lambda}} \left[1 - \left(\frac{\sqrt{|A|}}{\sqrt{|A+C|}} - \frac{\sqrt{|B|}}{\sqrt{|B+C|}} \right)^2 \right]^{\min\{\lambda, 1-\lambda\}}.$$

定理 8(改进 2) 设 A, B, C 和 D 是 4 个实的 n 阶正定矩阵, $0 \leq \lambda \leq 1$, 则有

$$\frac{1}{|\lambda A + (1-\lambda)B|} \leq \frac{1}{|A|^\lambda |B|^{1-\lambda}} (1 - \delta(A, B, C, D))^{2 \min\{\lambda, 1-\lambda\}},$$

其中 $|C| = \pi^n$,

$$\delta(A, B, C, D) = \left[\frac{|A|^{\frac{1}{4}}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}}} - \frac{|B|^{\frac{1}{4}}}{\left| \frac{1}{2}(B+C) \right|^{\frac{1}{2}}} \right]^2 \left[\sum_{k=0}^{m-1} \left(\frac{\sqrt{|A||B|}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}} \left| \frac{1}{2}(B+C) \right|^{\frac{1}{2}}} \right)^{\frac{k}{2}} \right] \\ + \frac{1}{2} \left(\frac{\sqrt{|A||B|}}{\left| \frac{1}{2}(A+C) \right|^{\frac{1}{2}} \left| \frac{1}{2}(B+C) \right|^{\frac{1}{2}}} \right)^{\frac{m}{2}} \left(\frac{\sqrt{|A|}}{\sqrt{|A+D|}} - \frac{\sqrt{|B|}}{\sqrt{|B+D|}} \right)^2.$$

Opial-华罗庚不等式 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数, $f(0) = 0$, 并且 $p > 0, q > 1$, 则有

$$\int_0^b |f|^p |f'|^q dx \leq \frac{qb^p}{p+q} \int_0^b |f'|^{p+q} dx.$$

相应的 Opial-华罗庚不等式的改进如下:

定理 9 设 $f(x)$ 为区间 $[0, h]$ 上的绝对连续函数, $f(0) = 0$, 并且 $p, q > 0$,

$p+q > 1$, 设 $1 - e(x) + e(y) \geq 0$, 并且 $e(x) = \frac{1}{2} \cos \frac{\pi x}{t}$, $s = \min\{1,$

$p+q-1\}$. 若 $0 \leq b_1 < b$, 则有

$$\int_{b_1}^b |f|^p |f'|^q dx + \frac{p(q-1)}{p+q} \left(\int_{b_1}^b x^{-q} |f|^{p+q} dx \right) + \frac{pq s}{2(p+q)} \pi(b_1, b) \\ \leq \frac{q}{p+q} \left(b^p \int_0^b |f'|^{p+q} dx - b_1^p \int_0^{b_1} |f'|^{p+q} dx \right),$$

其中

$$\pi(b_1, b) = \int_{b_1}^b t^{p-1} \int_0^t |f'|^{p+q} dx \left(\frac{1}{t} \int_0^t e(x) dx - \frac{\int_0^t e(x) |f'|^{p+q} dx}{\int_0^t |f'|^{p+q} dx} \right)^2 dt.$$

Ingham 不等式 设 a_k, b_k, λ 为任意实数, 记 $\|x\| = \sum_{k=0}^n |x_k|^2$, $S_{i,\lambda}(a, b) =$

$$\sum_{r,s=0}^n \frac{a_r b_s}{(r+s+\lambda)^i}, \text{ 则有}$$

$$|S_{1,\lambda}(a, b)|^2 \leq M^2(\lambda) \|a\| \cdot \|b\|,$$

其中

$$M(\lambda) = \begin{cases} \frac{\pi}{\sin \lambda \pi}, & 0 \leq \lambda \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < \lambda \leq 1. \end{cases}$$

相应的 Ingham 不等式的改进如下:

定理 10 (改进 1) 设 a_k, b_k, λ 为任意实数, 记 $T_i(\mathbf{a}, \mathbf{b}) = \sum_{l,m=0, l \neq m}^n \frac{a_l b_m}{(l-m)^i}$,

$$\|\mathbf{x}\| = \sum_{k=0}^n |x_k|^2, S_{i,\lambda}(\mathbf{a}, \mathbf{b}) = \sum_{r,s=0}^n \frac{a_r b_s}{(r+s+\lambda)^i}, \text{ 则有}$$

$$\begin{aligned} & \left| T_1(\mathbf{a}, \mathbf{b}) \right|^2 + \sin^2 \lambda \pi \left| S_{1,\lambda}(\mathbf{a}, \mathbf{b}) \cot \lambda \pi - \frac{1}{\pi} S_{2,\lambda}(\mathbf{a}, \mathbf{b}) \right|^2 \\ & \leq \pi^2 \left\{ \left[\|\mathbf{a}\| \|\mathbf{b}\| - \pi^{-2} \sin^2 \lambda \pi S_{1,\lambda}(\mathbf{a}, \bar{\mathbf{a}}) S_{1,\lambda}(\mathbf{b}, \bar{\mathbf{b}}) \right]^2 \right. \\ & \quad \left. - \frac{4}{\pi^4} (\tau_\lambda(\mathbf{a}, \mathbf{b}) + \tau_\lambda(\mathbf{b}, \mathbf{a}))^2 \right\}^{\frac{1}{2}}, \end{aligned}$$

其中

$$\begin{aligned} \tau_\lambda(\mathbf{a}, \mathbf{b}) = \sin \lambda \pi \left\{ \|\mathbf{b}\| \left[\frac{\pi}{3} S_{1,\lambda}(\mathbf{a}, \bar{\mathbf{a}}) + \cot \lambda \pi S_{2,\lambda}(\mathbf{a}, \bar{\mathbf{a}}) - \pi^{-2} S_{3,\lambda}(\mathbf{a}, \bar{\mathbf{a}}) \right] \right. \\ \left. - \pi^{-1} S_{1,\lambda}(\mathbf{b}, \bar{\mathbf{b}}) T_2(\mathbf{a}, \bar{\mathbf{a}}) \right\}. \end{aligned}$$

定理 11 (改进 2) 设 a_k, b_k, λ 为任意实数, 记 $T_i(\mathbf{a}, \mathbf{b}) = \sum_{l,m=0, l \neq m}^n \frac{a_l b_m}{(l-m)^i}$,

$$\|\mathbf{x}\| = \sum_{k=0}^n |x_k|^2, S_{i,\lambda}(\mathbf{a}, \mathbf{b}) = \sum_{r,s=0}^n \frac{a_r b_s}{(r+s+\lambda)^i}, \text{ 则有}$$

$$\begin{aligned} & \left| S_{1,\lambda}(a, b) \right|^2 + \left| \frac{T_1(a, b)}{\sin \lambda \pi} \right|^2 + \left| S_{1,\lambda}(a, b) \cot \lambda \pi - \frac{1}{\pi} S_{2,\lambda}(a, b) \right|^2 \\ & \leq \frac{\pi^2}{\sin^2 \lambda \pi} (\|a\| \|b\|). \end{aligned}$$

定理 12 (改进 3) 设 a_k, b_k, λ 为任意实数, 记 $T_i(\mathbf{a}, \mathbf{b}) = \sum_{l,m=0, l \neq m}^n \frac{a_l b_m}{(l-m)^i}$,

$$\|\mathbf{x}\| = \sum_{k=0}^n |x_k|^2, S_{i,\lambda}(\mathbf{a}, \mathbf{b}) = \sum_{r,s=0}^n \frac{a_r b_s}{(r+s+\lambda)^i}, \text{ 则有}$$

$$\begin{aligned} & \left(|T_1(\mathbf{a}, \mathbf{b})|^2 + |S_{1,1}(\mathbf{a}, \mathbf{b})|^2 \right)^2 + \left(\frac{4}{\pi} \right)^2 \left(S_{2,1}(\mathbf{a}, \bar{\mathbf{a}}) \|\mathbf{b}\| + S_{2,1}(\mathbf{b}, \bar{\mathbf{b}}) \|\mathbf{a}\| \right)^2 \\ & \leq \pi^4 (\|\mathbf{a}\| \|\mathbf{b}\|)^2. \end{aligned}$$

定理13 (改进4) 设 a_k, b_k, λ 为任意实数, 记 $T_i(\mathbf{a}, \mathbf{b}) = \sum_{l,m=0, l \neq m}^n \frac{a_l b_m}{(l-m)^i}$,

$$\begin{aligned} \|\mathbf{x}\| &= \sum_{k=0}^n |x_k|^2, \|\bar{\mathbf{x}}\| = \sum_{k=0}^n \frac{|x_k|}{k+1}, S_{i,\lambda}(\mathbf{a}, \mathbf{b}) = \sum_{r,s=0}^n \frac{a_r b_s}{(r+s+\lambda)^i}, \text{ 则有} \\ & |S_{1,\frac{1}{2}}(\mathbf{a}, \mathbf{b})|^2 + |T_1(\mathbf{a}, \mathbf{b})|^2 \\ & \leq \pi^2 \left\{ \left[\|\mathbf{a}\| \|\mathbf{b}\| - \frac{1}{\pi^4} S_{2,\frac{1}{2}}(\mathbf{a}, \bar{\mathbf{a}}) S_{2,\frac{1}{2}}(\mathbf{b}, \bar{\mathbf{b}}) \right]^2 \right. \\ & \quad \left. - \frac{1}{\pi^8} \left[\|\bar{\mathbf{a}}\| S_{2,\frac{1}{2}}(\mathbf{b}, \bar{\mathbf{b}}) + \|\bar{\mathbf{b}}\| S_{2,\frac{1}{2}}(\mathbf{a}, \bar{\mathbf{a}}) \right]^2 \right\}^{\frac{1}{2}}. \end{aligned}$$

定理14 (Hilbert 积分不等式的改进) 设 $f, g \geq 0$ 且 $f, g \in L^2(0, \infty)$, 则当

$1 - e(x) + e(y) \geq 0$ 时有

$$\begin{aligned} & \left(\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \right)^2 \\ & \leq \pi^2 \left[\left(\int_0^\infty f^2(x) dx \right)^2 - \left(\int_0^\infty f^2(x) \alpha(x) dx \right)^2 \right] \\ & \quad \cdot \left[\left(\int_0^\infty g^2(x) dx \right)^2 - \left(\int_0^\infty g^2(x) \alpha(x) dx \right)^2 \right], \end{aligned}$$

$$\text{其中 } \alpha(x) = \frac{2}{\pi} \int_0^\infty \frac{e(xt^2)}{1+t^2} dt - e(x).$$

Hardy 型不等式 设 $f(x) \in L^p(0, \infty)$, $p > 1$ 及 $f(x) \geq 0$, 则有

$$\int_0^\infty \left(\frac{\int_0^x f(t) dt}{x} \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) dx.$$

相应的 Hardy 型不等式的改进如下:

定理 15 (改进1) 设

$$(1) \quad 0 < \lambda \leq 1, \quad p, q > 1, \quad \lambda = 2 - \frac{1}{p} - \frac{1}{q};$$

(2) $f(x), g(x) \geq 0, x \in (0, \infty), F(y) \geq 0, y \geq 0$, 以及对每一正数 $h, \int_0^h g(x)dx$ 存在;

(3) 设函数 $e(x)$ 定义在 $[0, \infty)$ 上, 并且 $1 - e(x) + e(y) \geq 0$.

令 $q' = \frac{q}{q-1}$ 及

$$G_1(x) = \frac{g^{\frac{1}{q'}}(x) \int_0^x g(t) \left(\int_0^t g(v)dv \right)^{-\alpha\lambda} F^{\frac{1}{p}}(f(t))dt}{\left(\int_0^x g(t)dt \right)^{(1-\alpha)\lambda}},$$

$$G_2(x) = \frac{g^{\frac{1}{\lambda q'}}(x) \int_0^x g(t) \left(\int_0^t g(v)dv \right)^{-\alpha} F^{\frac{1}{\lambda p'}}(f(t))dt}{\left(\int_0^x g(t)dt \right)^{1-\alpha}}.$$

$H_1 = \left(g(x)F(f(x)) \right)^{\frac{1}{\lambda q'}}$. 若 $\alpha < 1 - \frac{1}{\lambda q'}$ 和 $H_1 \in L^{\lambda q'}(0, \infty)$, 则

$$\int_0^\infty G_1^{q'}(x)dx \leq \left(\frac{q'\lambda}{q'\lambda(1-\alpha)-1} \right)^{q'\lambda} \left(\int_0^\infty g(x)F(f(x))dx \right)^{q'(1-\lambda)+1} \cdot (1 - R^2(G_2, H_1))^{\theta(q'\lambda)},$$

$$\text{其中, } \theta(x) = \begin{cases} \frac{1}{2}, & x > 2, \\ \frac{1}{2}(x-1), & 1 < x \leq 2, \end{cases}$$

$$R(G_2, H_1) = \frac{\int_0^\infty G_2^{q'\lambda}(x)e(x)dx}{\int_0^\infty G_2^{q'\lambda}(x)dx} - \frac{\int_0^\infty H_1^{q'\lambda}(x)e(x)dx}{\int_0^\infty H_1^{q'\lambda}(x)dx}.$$

定理 16 (改进 2) 设

(1) $p > 1, r > 1;$

(2) $f(x), g(x) \geq 0, x \in (0, \infty), F(y) \geq 0, y \geq 0$ 及对每一正数 h , $\int_0^h g(x)dx$ 存在;

(3) 设函数 $e(x)$ 定义在 $[0, \infty)$ 上, 并且 $1 - e(x) + e(y) \geq 0$.
令

$$G_3(x) = \frac{g^{\frac{1}{p}}(x) \int_0^x g(t) F^{\frac{1}{p}}(f(t)) dt}{\left(\int_0^x g(t) dt \right)^{\frac{r}{p}}},$$

$$H_2(x) = g^{\frac{1}{p}}(x) \left(\int_0^x g(t) dt \right)^{\frac{p-r}{p}} F^{\frac{1}{p}}(f(x)).$$

若 $H_2 \in L^p(0, \infty)$, 则

$$\int_0^\infty G_3^p(x) dx \leq \left(\frac{p}{r-1} \right)^p \int_0^\infty H_2^p(x) dx (1 - R^2(G_3, H_2))^{\theta(p)},$$

$$\text{其中, } \theta(x) = \begin{cases} \frac{1}{2}, & x > 2, \\ \frac{1}{2}(x-1), & 1 < x \leq 2, \end{cases}$$

$$R(G_2, H_1) = \frac{\int_0^\infty G_2^{q'\lambda}(x) e(x) dx}{\int_0^\infty G_2^{q'\lambda}(x) dx} - \frac{\int_0^\infty H_1^{q'\lambda}(x) e'(x) dx}{\int_0^\infty H_1^{q'\lambda}(x) dx}.$$

Hardy-Littlewood-Polya 不等式 设 $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0, f \in$

$L^p(0, \infty), g \in L^q(0, \infty)$. 又设 $K(x, y) (\geq 0)$ 为齐负一次式, 并且设

$$\int_0^\infty K(x, 1) x^{-\frac{1}{p}} dx = \int_0^\infty K(1, y) y^{-\frac{1}{q}} dy = k.$$

则有

$$\int_0^\infty \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy \leq k^p \int_0^\infty f^p(x) dx,$$

以及

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy \\ & \leq k \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}}. \end{aligned}$$

相应的 Hardy-Littlewood-Polya 不等式的改进如下:

定理 17 (改进 1) 设 $p > 1$, $0 < \lambda \leq 1$ 及 $f(x) (\geq 0) \in L^p(0, \infty)$. 又设

$K(x, y) \geq 0$ 及 $(K(x, y))^{\frac{1}{\lambda}}$ 为齐负一次式. 若有 $q > 1$ 使 $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$

及

$$\int_0^\infty \left(K(x, 1) x^{-\frac{1}{q'}} \right)^{\frac{1}{\lambda}} dx = k, \quad q' = \frac{q}{q-1},$$

则

$$\int_0^\infty \left(\int_0^\infty K(x, y) f(x) dx \right)^{q'} dy \leq k^{\lambda q'} \left(\int_0^\infty f^p(x) dx \right)^{(1-\lambda)q'+1}.$$

定理 18 (改进 2) 设 $p > 1$, $0 < \lambda \leq 1$ 及 $f(x), g(x) (\geq 0) \in L^p(0, \infty)$,

$1 + e(x) - e(y) \geq 0$ 对 $x, y \in (0, \infty)$ 成立. 设 $K(x, y) \geq 0$ 及 $(K(x, y))^{\frac{1}{\lambda}}$

为齐负一次式. 又设 $q > 1$ 使 $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$ 且

$$\int_0^\infty \left(K(x, 1) x^{-\frac{1}{q'}} \right)^{\frac{1}{\lambda}} dx = k, \quad q' = \frac{q}{q-1},$$

则有

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy \\ & \leq k^\lambda \left(\int_0^\infty f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(y) dy \right)^{\frac{1}{q}} (1 - R^2(f, g))^{\frac{1}{2}\rho(q)}, \end{aligned}$$

其中 $\rho(q) = \min \left\{ \frac{1}{q}, \frac{1}{q'} \right\}$,

$$kE(x) = \int_0^\infty e\left(\frac{\omega}{x}\right) \left(K(\omega, 1) \omega^{-\frac{1}{q'}} \right)^{\frac{1}{\lambda}} d\omega,$$

$$R(f, g) = \frac{\int_0^\infty g^q(y)e(y)dy}{\int_0^\infty g^q(y)dy} - \frac{\int_0^\infty f^p(x)E(x)dx}{\int_0^\infty f^p(x)dx}.$$

定理 19 (改进 3) 设 $c > 0$, $p > 1$, $0 < \lambda \leq 1$ 及 $f(x), g(x) (\geq 0) \in L^p(0, \infty)$,

$1 + e(x) - e(y) \geq 0$ 对 $x, y \in (0, \infty)$ 成立. 设 $K(x, y) = (x^c + y^c)^{-\frac{\lambda}{c}}$,

$q > 1$ 使 $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$ 且 $q' = \frac{q}{q-1}$, 则有

$$\begin{aligned} & \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x^c + y^c)^{\frac{\lambda}{c}}} dx dy \\ & \leq c^{-\lambda} B^\lambda \left(\frac{1}{c\lambda p'}, \frac{1}{c\lambda q'} \right) \|f\|_p \|g\|_q (1 - R^2(f, g))^{\frac{1}{2}m(q)}, \end{aligned}$$

其中 $m(q) = \min \left\{ \frac{1}{q}, \frac{1}{q'} \right\}$, $B(a, b)$ 为 Beta 函数,

$$\begin{aligned} R(f, g) &= \frac{(g^q, e)}{\|g\|_q^q} - \frac{(f^p, E)}{\|f\|_p^p}, \\ c^{-1} B \left(\frac{1}{c\lambda p'}, \frac{1}{c\lambda q'} \right) E(x) &= \int_0^\infty \frac{1}{(\omega^c + 1)^{\frac{1}{c}}} \omega^{-\frac{1}{\lambda q'}} e \left(\frac{x}{\omega} \right) d\omega. \end{aligned}$$

定理 20 (改进 4) 同定理 19 所设, 但 $p = q$, $e(x) = \frac{1}{2} \cos \sqrt{x}$, $E(x) = \frac{1}{2} e^{-\sqrt{x}}$,

则

$$\begin{aligned} I(f) &= \int_0^\infty \int_0^\infty \frac{1}{(x+y)^\lambda} f(x)f(y) dx dy \\ &\leq \pi^\lambda \|f\|_p^2 \left[1 - \frac{1}{4} \left(\frac{\int_0^\infty f^q(x)(\cos \sqrt{x} - e^{-\sqrt{x}}) dx}{\|f\|_p^p} \right)^2 \right]^{\frac{1}{p}}. \end{aligned}$$

定理 21 (改进 5) 设 $1 + e(x) - e(y) \geq 0$ 对 $x, y \in (0, \infty)$ 成立, $\lambda > 0$,

$p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f, g \geq 0$. 又设 $K(x, y) \geq 0$, $|K(x, y)|^{\frac{1}{\lambda}}$ 为齐负一

次式. 若记

$$F_\lambda(x) = x^{\frac{1-\lambda}{q}} f(x) \in L^p(0, \infty), \quad G_\lambda(y) = y^{\frac{1-\lambda}{p}} g(y) \in L^p(0, \infty),$$

$$R(F_\lambda, G_\lambda) = \frac{(G_\lambda^q, e)}{\|G_\lambda\|_q^q} - \frac{(F_\lambda^p, E)}{\|F_\lambda\|_p^p}, \quad \int_0^\infty K(\omega, 1) \omega^{\frac{\lambda}{q}-1} d\omega = k,$$

$$\rho(p) = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad kE(x) = \int_0^\infty K(\omega, 1) e\left(\frac{x}{\omega}\right) \omega^{\frac{\lambda}{q}-1} d\omega.$$

则有

$$\int_0^\infty y^{\lambda-1} \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy \leq k \int_0^\infty x^{(p-1)(1-\lambda)} f^p(x) dx,$$

以及

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy \\ & \leq k \|F_\lambda\|_p \|G_\lambda\|_q (1 - R^2(F_\lambda, G_\lambda))^{\frac{\rho(p)}{2}}. \end{aligned}$$

定理 22 (改进 6) 设 $1 + e(x) - e(y) \geq 0$ 对 $x, y \in (0, \infty)$ 成立, $\lambda > 0$,

$p, q > 1, \frac{1}{p} + \frac{1}{q} = 1, f, g \geq 0$. 又设 $K(x, y) = \frac{1}{(x^c + y^c)^{\frac{\lambda}{c}}}$, $c > 0$. 若记

$$F_\lambda(x) = x^{\frac{1-\lambda}{q}} f(x) \in L^p(0, \infty), \quad G_\lambda(y) = y^{\frac{1-\lambda}{p}} g(y) \in L^p(0, \infty),$$

$$R(F_\lambda, G_\lambda) = \frac{(G_\lambda^q, e)}{\|G_\lambda\|_q^q} - \frac{(F_\lambda^p, E)}{\|F_\lambda\|_p^p}, \quad \int_0^\infty K(\omega, 1) \omega^{\frac{\lambda}{q}-1} d\omega = k,$$

$$\rho(p) = \min \left\{ \frac{1}{p}, \frac{1}{q} \right\}, \quad kE(x) = \int_0^\infty K(\omega, 1) e\left(\frac{x}{\omega}\right) \omega^{\frac{\lambda}{q}-1} d\omega.$$

则有

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy \\ & \leq \frac{1}{c} B\left(\frac{\lambda}{cp}, \frac{\lambda}{cq}\right) \|F_\lambda\|_p \|G_\lambda\|_q (1 - R^2(F_\lambda, G_\lambda))^{\frac{\rho(p)}{2}}. \end{aligned}$$

定理 23 (改进 7) 如定理 22 所设, 取 $p = 2$, 记

$$I(f, g) = \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\lambda + y^\lambda} dx dy.$$

则有

$$I(f, g) \leq \frac{\pi}{\lambda} \|F_\lambda\|_2 \|G_\lambda\|_2 (1 - R_i^2(F_\lambda, G_\lambda))^{\frac{1}{4}} \cdot (1 - R_i^2(G_\lambda, G_\lambda))^{\frac{1}{4}}, \quad i = 1, 2,$$

其中

$$R_i(h, h) = \frac{\frac{1}{2} \int_0^\infty h^2(y) (\cos \sqrt{y^\lambda} - e^{-\sqrt{y^\lambda}}) dy}{\|h\|_2^2}, \quad h = F_\lambda, G_\lambda.$$

定理 24 (改进 8) 设 $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $f(x) \geq 0$, $g(y) \geq 0$ 及 $x^{\frac{1-\lambda}{p}} f(x) \in L^p(0, \infty)$, $y^{\frac{1-\lambda}{q}} g(y) \in L^p(0, \infty)$, 并且对任意的 $x, y \in (0, \infty)$, 都有 $1 + e(x) - e(y) \geq 0$. 又设 $K(x, y) \geq 0$, $K^{\frac{1}{\lambda}}(x, y)$ 为关于 x, y 的齐负一次式. 若记

$$\int_0^\infty K(x, 1) x^{\frac{\lambda-2}{p}} dx = k,$$

$$R(f, g) = \frac{\int_0^\infty y^{1-\lambda} e(y) g^p(y) dy}{\int_0^\infty y^{1-\lambda} g^p(y) dy} - \frac{\int_0^\infty x^{1-\lambda} f^p(x) E(x) dx}{\int_0^\infty x^{1-\lambda} f^p(x) dx},$$

$$kE(x) = \int_0^\infty e\left(\frac{x}{\omega}\right) K(\omega, 1) \omega^{\frac{\lambda-2}{p}} d\omega, \quad \theta(p) = \begin{cases} \frac{1}{2p}, & p \geq q, \\ \frac{1}{2q}, & q > p, \end{cases}$$

则有

$$\int_0^\infty y^{\frac{(\lambda-1)p}{q}} \left(\int_0^\infty K(x, y) f(x) dx \right)^p dy \leq k^p \int_0^\infty x^{1-\lambda} f^p(x) dx,$$

以及

$$\begin{aligned} & \int_0^\infty \int_0^\infty K(x, y) f(x) g(y) dx dy \\ & \leq k \left(\int_0^\infty x^{1-\lambda} f^p(x) dx \right)^{\frac{1}{p}} \left(\int_0^\infty y^{1-\lambda} g^q(y) dy \right)^{\frac{1}{q}} (1 - R^2(f, g))^{\theta(p)}. \end{aligned}$$

定理 25 (改进 9) 设 $1 - \alpha\lambda + \lambda p + \frac{\lambda - 2}{p} > 0$, $f \geq 0$, $x^{\frac{1-\lambda}{p}} f \in L^p(0, \infty)$,

则有

$$\begin{aligned} & \int_0^\infty y^{\frac{(\lambda-1)p}{q}} \left(\int_0^y \frac{y^{(\alpha-1)\lambda}}{x^{\alpha\lambda}} f(x) dx \right)^p dy \\ & \leq \frac{p^p}{[p - \lambda(\alpha - 1)p + \lambda - 2]^p} \int_0^\infty x^{1-\lambda} f^p(x) dx. \end{aligned}$$

定理 26 (改进 10) 设 $\lambda - 2 + \min\{p, q\} > 0$, $f(x) \geq 0$, $x^{\frac{1-\lambda}{p}} f \in L^p(0, \infty)$,

则有

$$\begin{aligned} & \int_0^\infty y^{\frac{(\lambda-1)p}{q}} \left[\int_0^\infty f(x)(x+y)^{-\lambda} dx \right]^p dy \\ & \leq B^p \left(1 + \frac{\lambda-2}{p}, 1 + \frac{\lambda-2}{p} \right) \int_0^\infty x^{1-\lambda} f^p(x) dx. \end{aligned}$$

参考文献

- [1] 胡克. 一个不等式及其若干应用. 中国科学, 1981, 2: 141-148.
- [2] 胡克. 论 Nagy-Carlson 型不等式. 江西师范大学学报, 1993, 17 (2): 89-91.
- [3] 胡克. 关于 Beekensbach 不等式. 江西师范大学学报, 1994, 18 (2): 140-141.
- [4] 胡克. 论 Hölder 不等式. 江西师范大学学报, 1994, 18 (3): 205-207.
- [5] 胡克. 关于 Hardy-Littlewood-Polya 不等式及其应用. 数学年刊, 1994, 15 (5): 524-527.
- [6] 胡克. 论 Opial-Beesack 不等式. 江西师范大学学报, 1995, 19 (1): 23-26.
- [7] 胡克. 关于 Minkowski 不等式. 江西师范大学学报, 1995, 19 (4): 285-287.
- [8] 胡克. 伪平均不等式的改进与推广. 抚州师专学报, 1996, 1 (1): 1-3.
- [9] 胡克. 论 Opial-华罗庚型积分不等式. 数学年刊, 1996, 17 (5): 517-518.
- [10] 胡克. 论一个不等式及其若干应用. 数学物理学报, 1998, 18 (2): 192-199.
- [11] 胡克. 单叶函数的若干问题. 武汉: 武汉大学出版社, 2001.
- [12] 胡克. 一个新的不等式及若干应用. 数学理论与应用, 2002, 22 (2): 1-6.
- [13] 胡克. 解析不等式的若干问题. 武汉: 武汉大学出版社, 2003.
- [14] 胡克. 关于 Aczél-Popoviciu-Vasic 不等式. 江西师范大学学报,

- 2006, 30 (2): 158-160.
- [15] 匡继昌. 常用不等式(第四版). 济南: 山东科学技术出版社, 2010.
- [16] Aczél J. Some general methods in the theory of functional equations in one variable. *New applications of functional equations*, *Uspehi. Mat. Nauk (N. S.)* (in Russian), 1956, 11: 3-68.
- [17] Beckenbach E F, A class of mean-value functions. *The American Mathematical Monthly*, 1950, 57: 1-6.
- [18] Beckenbach E F. On Hölder's inequality. *Journal of Mathematical Analysis and Applications*, 1966, 15: 21-29.
- [19] Beckenbach E F, Bellman R. *Inequalities*. Berlin: Springer, 1983.
- [20] Hölder O. Über einen mittelwertsatz. *Göttinger Nachrichten*, 1889, 2: 38-47.
- [21] Rogers L J. An extension of a certain theorem in inequalities. *Messenger of Math.*, 1888, 17: 145-150.
- [22] Hardy G H, Littlewood J E, Polya G. *Inequalities*. Cambridge: Cambridge University Press, 1952.
- [23] Hao Z-C. Note on the inequality of the arithmetic and geometric means. *Pacific J. Math.*, 1990, 143: 43-46.
- [24] Hu K. On an inequality and its applications. *Scientia Sinica*, 1981, 24 (8): 1047-1055.
- [25] Hu K. On Hilbert's inequality. *Chinese Annals of Mathematics*, 1992, 13 (1): 35-39.
- [26] Hu K. On Hilbert type inequality and its application. *Journal of Jiangxi Normal University*, 2001, 25 (2): 115-120.
- [27] Hu K. On Polya-Szegő inequalities. *Analysis in theory and applications*, 2005, 21 (4): 395-398.
- [28] Hu K. On an inequality and its applications. *Journal of Jiangxi Normal University*, 1994, 18 (4): 330-333.
- [29] Mitrinović D S, Pečarić J E, Fink A M. *Classical and New Inequalities in Analysis*. Kluwer, Dordrecht, 1993.
- [30] Mitrinović D S, Vasić P M. *Analytic Inequalities*. New York: Springer, 1970.

- [31] Popoviciu T. On an inequality. *Gaz. Mat. Fiz. Ser. A* (in Romanian), 1959, 11: 451-461.
- [32] Tian J-F. Extension of Hu Ke's inequality and its applications. *Journal of Inequalities and Applications*, 2011, vol. 2011, article 77.
- [33] Tian J. Reversed version of a generalized sharp Hölder's inequality and its applications. *Information Sciences*, 2012, 201: 61-69.
- [34] Tian J-F. Reversed version of a generalized Aczel's inequality and its application. *Journal of Inequalities and Applications*, 2012, Vol. 2012, article 202.
- [35] Tian J-F. Property of a Hölder-type inequality and its application. *Mathematical Inequalities & Applications*, 2013, 16 (3): 831-841.
- [36] Tian J, Hu X-M. A new reversed version of a generalized sharp Hölder's inequality and its applications. *Abstract and Applied Analysis*, 2013, vol. 2013, Article ID 901824, 9 pages.
- [37] Tian J, Zhou Y-X. Refinements of Hardy-type inequalities, *Abstract and Applied Analysis*. 2013, vol. 2013, Article ID 727923, 7 pages.
- [38] Vasić P M, Pečarić J E. On Hölder and some related inequalities. *Mathematica Rev. D'Anal. Num. Th. L'Approx.*, 1982, 25: 95-103.
- [39] Wang C-L. Characteristics of nonlinear positive functional and their applications. *Journal of Mathematical Analysis and Applications*, 1983, 95: 564-574.
- [40] Wu S, Debnath L. Generalizations of Aczél's inequality and Popoviciu's inequality, *Indian Journal of Pure and Applied Mathematics*, 2005, 36 (2): 49-62.
- [41] Wu S. Generalization of a sharp Hölder's inequality and its application. *Journal of Mathematical Analysis and Applications*, 2007, 332 (1): 741-750.
- [42] Wu S. A new sharpened and generalized version of Hölder's inequality and its applications. *Applied Mathematics and Computation*, 2008, 197 (2): 708-714.

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5. 6Wang G- L□ □ □ □ Beckenbach□ □ □ □ □ □ □ □ □ □ □ □

5. 7□ □ □ Beckenbach□ □ □ □ □ □ □ □

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